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**Contributions to optimal phase approximation
in lowpass and bandpass systems**

**(Beiträge zur optimalen Phasenapproximation
in Tiefpass-und Bandpaßsystemen)**

Shehata Mahmoud Hussein

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Contributions to optimal phase approximation in lowpass and bandpass systems

(Beiträge zur optimalen Phasenapproximation in Tiefpaß- und Bandpaßsystemen)

DISSERTATION
zur Erlangung des Grades eines
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Summary

The equiripple approximation of the delay time of an allpass-transmission function has been worked out considering it as an optimal approximation by ULBRICH and PILOTY [10] and also by ABELE [11]. The Phase function is also approximated by HUMPHREYS [12]. In our work, we have considered the ideal (linear) phase characteristic according to the following criteria:

$$a) \quad B(\omega) = T_m \cdot \omega + B_0 \quad (\text{TYPE III})$$

$$b) \quad \frac{\partial B(\omega)}{\partial \omega} = T(\omega) = T_m \quad (\text{TYPE I})$$

$$c) \quad \frac{\partial^2 B(\omega)}{\partial \omega^2} = V(\omega) = 0 \quad (\text{TYPE II}) \quad ,$$

where $B(\omega)$ is the phase function, $T(\omega)$ the group delay time and ω the angular frequency. B_0 and T_m are constants. The main part of this thesis is the equiripple approximation of the functions mentioned above together with a quantitative comparison of the results. The approximation is worked out for lowpass and bandpass intervals. The work is subdivided into four chapters which are summarized as follows:

CHAPTER I

The ideal phase characteristic and its important properties are described. The general approximation criteria are briefly mentioned and reference is made to the relevant literature. The main properties of the methods used in similar types of work are summarized and shown in a table.

CHAPTER II

To solve the equiripple approximation a numerical method is devised.

The method and the computer flow-diagram for TYPE I are described in detail, whereas for TYPE II and TYPE III only the differences in computer programs are mentioned. The solution of all three types are in tabular form, given in appendices for different filter degree and ripple magnitude about 30% of the mean delay time.

A series of examples are plotted throughout the work.

Furthermore, other solutions of TYPE I and TYPE II, with a constraint at the origin, are discussed. The solution for TYPE III is worked out for $B_0 = 0$ and the computer program is subsequently extended to solve it for $B_0 \neq 0$.

The approximation of TYPE III for $B_0 \neq 0$ is denoted as "TYPE III-general", because in this case the approximation interval does not start from the origin. Hereafter the approach to the bandpass approximation is clear.

CHAPTER III

A quantitative comparison between the parameters of lowpass filters is worked out in frequency and time domain to find out the best criteria from a), b), and c). The filter parameter definitions are stated and then calculated by two different methods. The results are given in tables.

The results show that the criterion (a) TYPE III has the best properties in both frequency and time domain.

CHAPTER IV

Finally the bandpass approximation is worked out through variation of B_0 which determines the relative bandwidth for different ripple magnitudes and filter degrees.

The solutions of different selected relative bandwidths, different ripple magnitudes and filter degrees are compiled in the appendices. A detailed example is given to demonstrate how the tables are used (denormalization) and how the elements of a ladder filter are calculated. This filter example is implemented and the calculated and measured phase characteristics are shown in plotted form.

Zusammenfassung

Die Approximation der Laufzeit von Allpaßübertragungsfunktionen mit gleicher Welligkeit (Equiripple) wurde von Ulbrich und Piloty [10] sowie Abele [11] durchgeführt und als optimale Approximation betrachtet. Ebenso wurde die Phasenfunktion auch von Humphreys [12] approximiert. In dieser Arbeit nun wird der ideale (gerade) Phasenverlauf nach folgenden drei Kriterien ausgewählt:

$$\text{a) } B(\omega) = T_m \cdot \omega + B_0 \quad (\text{TYP III})$$

$$\text{b) } \frac{\partial B(\omega)}{\partial \omega} = T(\omega) = T_m \quad (\text{TYP I})$$

$$\text{c) } \frac{\partial^2 B(\omega)}{\partial \omega^2} = V(\omega) = 0 \quad (\text{TYP II})$$

$B(\omega)$ ist die Phasenfunktion, $T(\omega)$ die Gruppenlaufzeit, ω die Kreisfrequenz, B_0 und T_m sind Konstanten.

Der Schwerpunkt dieser Dissertation liegt in der Equiripple-Approximationen der oben genannten drei Funktionen und in dem Vergleich der Resultate untereinander. Die Approximationen sind für Tiefpässe und Bandpässe ausgeführt.

Die vorliegende Arbeit ist in vier Kapitel, wie folgt aufgeteilt:

Kapitel I:

In diesem Kapitel ist die ideale Phasencharakteristik mit ihren wichtigsten Eigenschaften der Phasenfunktion angegeben. Die allgemeinen Approximationskriterien sind kurz aufgezeigt; dabei sind die wichtigsten Literaturquellen aufgeführt. Die früheren Arbeiten auf diesem Gebiet werden erwähnt und die Eigenschaften dieser Methoden in einer Tabelle angegeben.

Kapitel II:

Es wurden für die Lösung der Equiripple-Approximation numerische Verfahren entwickelt, die in diesem Kapitel aufgezeigt sind. Für TYP I sind die Verfahren und das Flußdiagramm des Rechen-

programms aufgeführt; die wichtigsten Änderungen im Rechenprogramm für TYP II und III sind gezeigt. Für alle drei Typen sind die Lösungen in Tabellenform im Appendix für verschiedene Filtergrade und für eine Rippelhöhe bis 30 % der mittleren Laufzeit angegeben. Für eine Reihe von Beispielen ist das Ergebnis in graphischer Form zusätzlich dargestellt.

Weiterhin sind die möglichen Lösungen der TYPEN I und II mit vorgeschriebenen Bedingungen bei der Frequenz Null aufgeführt. Nach der Lösung für den TYP III bei $B_0 = 0$ wurde das Programm noch für den Fall $B_0 \neq 0$ erweitert. Die einschränkenden Bedingungen für das Arbeiten mit $B_0 \neq 0$ wurden angegeben..

Die Approximation des Typs III bei $B_0 \neq 0$ ist als "TYPE III-general" bezeichnet; dieser Name wurde gewählt, weil nun das Approximationsintervall nicht mehr bei Null anfängt. Damit wurde der Weg zu den Bandpassapproximation gefunden.

Kapitel III:

Um das beste Kriterium gemäß a), b), und c) herauszufinden, wurden in diesem Kapitel die Eigenschaften von Tiefpässen im Frequenz- und Zeitbereich berechnet und miteinander verglichen. Für die Eigenschaften der zu untersuchenden Filter wurden die benützten Definitionen angegeben und das Verhalten mittels zweier Methoden berechnet. Die Ergebnisse wurden miteinander verglichen und in Tabellenform dargestellt.

Die Resultate zeigen, daß das Kriterium (a) (TYP III) sowohl im Frequenz- wie auch im Zeitbereich die besten Eigenschaften besitzt.

Kapitel 4:

In diesem Kapitel ist die Bandpaßapproximation durchgeführt.

Zusammenfassungen der früheren Arbeiten sind einleitend angegeben. Die Bandpaßapproximation wurde durch Variieren von B_0 gewonnen; d.h. B_0 bestimmt die relative Bandbreite für verschiedene Rippelhöhen bei gegebenem Filtergrad.

Für die praktische Anwendung sind die Lösungen für verschiedene ausgewählte relative Bandbreiten, verschiedene Rippelhöhen und Filtergrade im Appendix angegeben. Ebenso wurde ein Beispiel zur Handhabung der Tabellen (Entnormierung) und die Berechnung der Elemente der Ladder-filter gezeigt. Dieses Filterbeispiel wurde realisiert; der gerechnete und der praktische Phasenverlauf sind graphisch dargestellt.

CHAPTER I (Introduction & Review)

1.1 Introduction

This chapter is concerned with the fundamental theories of phase characteristics and the methods of approximation required in this thesis. The theories and the important relations are briefly described and, according to their importance, some of them are explained. Important references for each case are mentioned, and those relations which are not explained in the literature are discussed in detail. Initially, the main characteristics of transmission channel are discussed and the ideal characteristics are deduced. The phase and group delay time of a transmission channel are explained and their effect on the transmitted signal is also discussed. The phase function $B(\omega)$, delay time $T(\omega)$ and the differentiation of the delay time $V(\omega)$ are given special attention because they are directly related to the phase approximation problem. The three main approximation criteria, maximally flat, least mean square error and equal ripple are stated and their inter-relation is discussed. The general form of these three criteria for all types of approximation is given briefly as required. In addition a short review of adopted techniques to approximate the phase characteristic (delay time), as well as a simple comparison of these techniques, is given at the end of this chapter.

1.2 The main characteristics of transmission channels

The transmission channel is generally represented by a closed box having input and output terminals Fig. (1-1). Considering it as a two-port network, then the input and output signals in the frequency domain are related [1] by

$$Y(j\omega) = G(\omega)e^{-jB(\omega)}X(j\omega) \quad , \quad (1.1)$$

where $X(j\omega)$ and $Y(j\omega)$ are the transformation of input $x(t)$ and output $y(t)$ respectively.

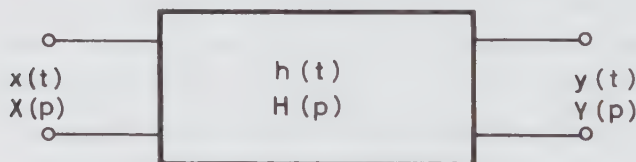


Fig. (1-1) Two-port network represents general transmission channel.

The ratio between $Y(j\omega)$ and $X(j\omega)$ is defined as the transfer function and is denoted by $H(j\omega)$ which is given by

$$H(j\omega) = G(\omega)e^{-jB(\omega)} \quad , \quad (1.2)$$

where

$$G(\omega) = |H(j\omega)| \quad (1.3)$$

and

$$B(\omega) = -\arg(H(j\omega)) \quad .$$

$G(\omega)$ contains the channel attenuation characteristic and $B(\omega)$ the phase characteristic. In general such transfer functions, Eq. (1.2), are written with a complex variable p , instead of $j\omega$, and for finite lumped channels are expressed by a rational function

$$H(p) = \frac{b_m p^m + b_{m-1} p^{m-1} + \dots + b_0}{a_n p^n + a_{n-1} p^{n-1} + \dots + a_0} \quad (1.4)$$

The polynomial in the denominator is a Hurwitz polynomial and in general $m \leq n$.

1.2.1 The ideal transmission channel

It is possible to define an ideal transmission channel by selecting the parameters of $H(p)$ such that

$$y(t) = x(t) \quad (1.5)$$

In practice a constant gain and a delay of the transmitted signal will not produce any unacceptable distortion. Therefore, the output signal of an ideal channel can be written as

$$y(t) = k x(t - T_m) \quad (1.6)$$

The frequency transformation of Eq. (1.6) is

$$Y(j\omega) = k e^{-j\omega T_m} X(j\omega) \quad (1.7)$$

Comparing Eq. (1.7) and (1.1), the ideal transfer function of a transmission channel is given as

$$H(j\omega) = k e^{-j\omega T_m} \quad (1.8)$$

and

$$\hat{G}(\omega) = k \quad (1.9)$$

$$\phi(\omega) = \omega T_m \quad (1.10)$$

Eq. (1.9) and (1.10) show that the ideal transmission channel in the transmission band has a constant attenuation and a linear phase function passing through the origin.

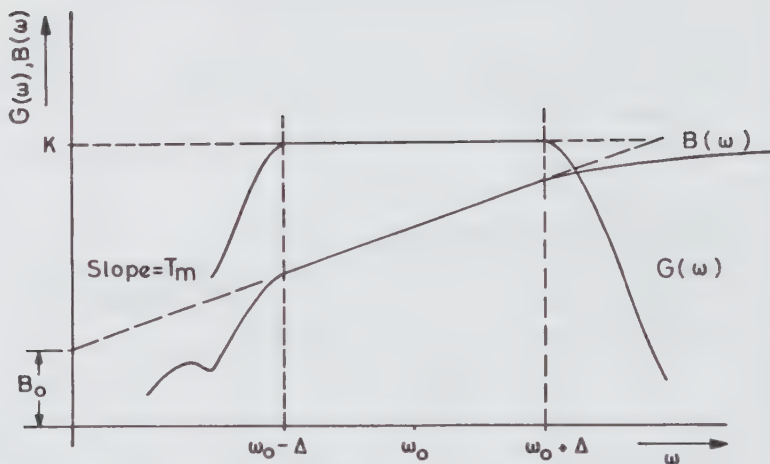


Fig. (1-2) Ideal attenuation $G(\omega)$ and phase $B(\omega)$ characteristics of a bandpass transmission channel

It will be shown in the next section, that for AM- or FM-signals the linear phase function (1.10) need not to pass through the origin. Fig. (1-2) shows the ideal attenuation and phase characteristics of a bandpass channel center about ω_0 . In this type of channel, in the bandpass interval $(\omega_0 - \Delta) < \omega < (\omega_0 + \Delta)$ the attenuation is a constant, Eq. (1.9), and the phase characteristic is

$$B(\omega) = T_m \omega + B_0, \quad (1.11)$$

where B_0 and T_m are constants.

$B(\omega)$ is an arbitrary straight line equation. The first derivative of Eq. (1.11) w.r.t. ω is

$$\frac{\partial B(\omega)}{\partial \omega} = T_m, \quad (1.12)$$

In general $\frac{\partial B(\omega)}{\partial \omega}$ is a function of ω , say $T(\omega)$, and it is sufficient to keep the phase characteristic in an ideal form if

$$T(\omega) = T_m \quad (1.13)$$

or

$$\frac{\partial T(\omega)}{\partial \omega} = 0 \quad (1.14)$$

Obviously Eq. (1.14) is the second derivative of Eq. (1.11) and its realization will also keep the phase characteristic in an ideal form. Therefore any equation from Eqs. (1.11), (1.13) and (1.14) will be sufficient to meet the ideal phase characteristic.

1.2.2 Phase delay time and group delay time

Though the delay time (phase-group) is one of the fundamental characteristics of the transmission channel, only few authors have discussed it [2] or given a mathematical analysis [3]. These results are summarised as follows

Consider a transmission channel described as

$$H(j\omega) = ke^{-jB(\omega)} \quad (1.15)$$

where the attenuation characteristic is kept constant and $B(\omega)$ is in the form of Eq. (1.11). If the input signal $x(t)$ has a general form of amplitude or frequency modulation of a carrier with a frequency ω_0 , i.e.

$$x(t) = \text{Re}\{z(t) e^{j\omega_0 t}\} \quad (1.16)$$

where $Z(j\omega) = 0$ for $|\omega| < \Delta$ Fig.(1-2), the output signal $y(t)$ is shown in [3] as

$$y(t) = K \text{Re}\{z(t-t_g) e^{j\omega_0(t-t_{ph})}\} \quad (1.17)$$

where

$$t_g = \frac{\partial B(\omega)}{\partial \omega} \quad , \quad (\text{group delay time}) \quad (1.18)$$

and

$$t_{ph} = \left. \frac{B(\omega)}{\omega} \right|_{\omega=\omega_0} \quad (\text{phase delay time}) \quad (1.19)$$

In equation (1.17) it can easily be seen that the carrier is delayed by t_{ph} (phase delay time) whereas the signal $z(t)$, which contains the transmitted information, is delayed by t_g (group delay time). Therefore, for an ideal channel, the group delay time Eq. (1.18) instead of the phase delay time Eq. (1.19) should be kept constant in the transmission band.

Therefore, the ideal channel requires

$$\frac{\partial B(\omega)}{\partial \omega} = \text{constant}$$

or

$$B(\omega) = T_m \omega + B_0, \quad (1.20)$$

where T_m and B_0 are constants.

Equation (1.20) is the general form of the ideal phase function Eq. (1.10) and is the same as Eq. (1.11).

1.3 The phase characteristic of transmission channels

The main part of this thesis is the approximation of the phase characteristic in an ideal manner. Therefore, the main characteristics of phase function, delay time^{††} and differentiation of the delay time are explained in the following review.

In the study of the phase characteristic the attenuation is kept constant. Therefore, if the numerator of Eq. (1.4) is even or odd, the corresponding all-pass transfer function $H_a(p)$ will be

$$H_a(p) = \frac{1 - a_1 p + a_2 p^2 - a_3 p^3 + \dots + (-1)^n a_n p^n}{1 + a_1 p + a_2 p^2 + a_3 p^3 + \dots + a_n p^n}. \quad (1.21)$$

In Eq. (1.21) the numerator and denominator have the same phase characteristics. Therefore, the general form of $H(p)$ for the

^{††}Henceforth "delay time" means group delay time.

study of the phase characteristic is

$$H(p) = \frac{k}{1+a_1p+a_2p^2+a_3p^3+\dots+a_np^n}, \quad (1.22)$$

where k is constant. This is an important form of the transfer function which will be used in the following work. If the denominator of $H(p)$ is arranged in an even term $M(\omega)$ and an odd term $N(\omega)$, then

$$H(j\omega) = \frac{k}{M(\omega)+jN(\omega)}, \quad (1.23)$$

where $M(\omega) = 1-a_2\omega^2+a_4\omega^4-\dots$

and $N(\omega) = a_1\omega-a_3\omega^3+a_5\omega^5-\dots$

The phase function $B(\omega)$ of eq. (1.23) is

$$\begin{aligned} B(\omega) &= -\arg\left(\frac{k}{M(\omega)+jN(\omega)}\right), \\ &= \tan^{-1}\left(\frac{N(\omega)}{M(\omega)}\right). \end{aligned} \quad (1.24)$$

Therefore, the delay time will be

$$T(\omega) = \frac{\partial B(\omega)}{\partial \omega}$$

or

$$T(\omega) = \frac{MN' - NM'}{M^2 + N^2}, \quad (1.25)$$

where M , N , M' and N' are the abbreviations for $M(\omega)$, $N(\omega)$,

$\frac{\partial M(\omega)}{\partial \omega}$ and $\frac{\partial N(\omega)}{\partial \omega}$ respectively. Another expression for $T(\omega)$

(Appendix I) is

$$T(\omega) = \sum_{i=1}^n \frac{-p_i}{(\omega^2 + p_i^2)}, \quad (1.26)$$

where p_i is the i -th pole of $H(p)$ in eq. (1.22).

This expression will be more useful in the coming chapters.

To complete the foregoing discussion some important properties of the delay time must be derived. These properties are basically simple but are of importance in the present work. Moreover, these properties have not been reviewed previously. They are given as follows:

Examining the general transfer function $H(p)$ of Eq. (1.22) and normalizing it to its amplitude $|H(p)|$, we get,

$$H_n(\omega) = \frac{H(p)}{|H(p)|} \bigg|_{p=j\omega} \quad (1.27)$$

$H_n(\omega)$ is the normalized transfer function. From Eq. (1.2) and (1.27) the normalized transfer function is

$$H_n(\omega) = e^{-jB(\omega)} \quad (1.28)$$

The derivatives on both sides w.r.t. ω will be

$$\frac{\partial H_n(\omega)}{\partial \omega} = -j \frac{\partial B(\omega)}{\partial \omega} e^{-jB(\omega)} \quad (1.29)$$

and

$$T(\omega) = j \frac{H'_n(\omega)}{H_n(\omega)} \quad (1.30)$$

where $H'_n(\omega) = \frac{\partial H_n(\omega)}{\partial \omega}$.

Eq. (1.30) gives the delay time in a compact form and it is derived from the transfer function $H(p)$ instead of the phase function $B(\omega)$. Considering for example

$$H(j\omega) = \frac{k}{M+jN} \quad (1.31)$$

where M and N are the abbreviations for $M(\omega)$ and $N(\omega)$ in Eq. (1.23) respectively.

then

$$H_n(\omega) = \frac{M - jN}{(M^2 + N^2)^{1/2}}, \quad (1.32)$$

and

$$\frac{H'_n(\omega)}{H_n(\omega)} = \frac{M' - jN'}{M - jN} - \frac{MM' + NN'}{M^2 + N^2}. \quad (1.33)$$

Substituting eq. (1.33) in eq. (1.30) then,

$$T(\omega) = \frac{MN' - NM'}{M^2 + N^2}, \quad (1.34)$$

which is the same as Eq. (1.25) derived in another manner. In this equation MN' , M^2 , N^2 , and NM' are even functions and therefore $T(\omega)$ will be an even rational function of ω .

Writing it again in another form

$$T(\omega) = \frac{R(\omega)}{Q(\omega)}, \quad (1.35)$$

where $R(\omega) = MN' - NM'$ and $Q(\omega) = M^2 + N^2$.

Multiplying eq. (1.35) by $Q(\omega)$ on both sides and then differentiating it w.r.t. ω gives

$$R'(\omega) = T(\omega)Q'(\omega) + T'(\omega)Q(\omega). \quad (1.36)$$

At a certain point, say $\omega = \omega_a$, where $T'(\omega) = 0$, $T(\omega)$ will be

$$T(\omega) = \left. \frac{R'(\omega)}{Q'(\omega)} \right|_{\omega = \omega_a} \quad (1.37)$$

Simplifying Eq. (1.35)

$$R = TQ \quad (1.38)$$

The n -th derivative on both sides of Eq. (1.38) w.r.t. ω is

$$R^{(n)} = TQ^{(n)} + C_1 T^{(1)} Q^{(n-1)} + C_2 T^{(2)} Q^{(n-2)} + \dots + C_n T^{(n)} Q, \quad (1.39)$$

where $R^{(i)}$, $T^{(i)}$ and $Q^{(i)}$ are the i -th derivatives of R , T and Q

respectively and $C_1, C_2, \dots, C_n$ are constants.

Eq. (1.39) shows at a certain point $\omega = \omega_b$, where all the derivatives of T up to the n -th one are equal to zero, i.e.

$$T^{(i)} = 0, \quad i=1, 2, 3, \dots, n$$

that the delay time will be

$$T(\omega) = \left. \frac{R(\omega)^{(n)}}{Q(\omega)^{(n)}} \right|_{\omega=\omega_b} . \quad (1.40)$$

Eq. (1.40) is important because it gives the delay time in a simple form under the condition that all the derivatives up to the n -th one vanish at a certain point which is the case for a maximally flat criterion which is discussed in the next section.

A mathematical relation similar to Eq. (1.37) is also found for the differentiation of the delay time, $V(\omega)$. From Eq. (1.36) $V(\omega)$ is:

$$V(\omega) = T'(\omega) = \frac{R'(\omega)}{Q(\omega)} - \frac{Q'(\omega)}{Q(\omega)} T(\omega) , \quad (1.41)$$

$$V(\omega) = \frac{C(\omega)}{Q(\omega)} , \quad (1.42)$$

where

$$C(\omega) = R'(\omega) - T(\omega) Q'(\omega) . \quad (1.43)$$

At a point, say $\omega = \omega_c$ where $V'(\omega) = 0$ and if the same procedure is carried out as for Eq. (1.35 ... 1.37), then Eq. (1.42) will be

$$V(\omega) = \left. \frac{C'(\omega)}{Q'(\omega)} \right|_{\omega=\omega_c} , \quad (1.44)$$

where, from Eq. (1.43)

$$C'(\omega) = R''(\omega) - T(\omega) Q''(\omega) - V(\omega) Q'(\omega) . \quad (1.45)$$

Substituting Eq. (1.45) in (1.44) gives

$$V(\omega) = \left. \frac{R''(\omega) - T(\omega)Q''(\omega)}{2Q'(\omega)} \right|_{\omega=\omega_0}, \quad (1.46)$$

where $R''(\omega)$ and $Q''(\omega)$ are the second derivative of $R(\omega)$, and $Q(\omega)$ respectively.

1.4. The approximation problem in general

In all cases a transmission channel is well defined by its transfer function. The transfer function $H(p)$ Eq. (1.22) is a rational function of order N , which contains a maximum of N unknown coefficients ($a_1, a_2 \dots, a_N$). For the sake of network realization, the denominator of $H(p)$ should be a Hurwitz polynomial.

Generally, there are many different methods for approximation and each of them is mainly dependent on what is called the approximation criterion. Certainly the problem of approximation is the adjustment of a set of parameters in a transfer function to approximate an ideal characteristic w. r. t. a specific criterion. The three main criteria, namely:

1. maximally flat criterion ,
2. least mean square error criterion ,
3. equal ripple criterion ,

are known in the approximation of transfer functions. The main properties of these three criteria are discussed in the following sections.

1.4.1 Maximally flat criterion

In this case the approximated function will be maximally flat at a certain point if at this point, all the derivatives of the given function are zero or, mathematically it will be

$$\left. \frac{\partial^m F(\omega, a_1, a_2, \dots, a_N)}{\partial \omega^m} \right|_{\omega=\omega_0} = 0, \quad (1.47)$$

where $F(\omega, a_1, a_2, \dots, a_N)$ is the function to be approximated, $m = 1, 2, \dots, N$,

and ω_0 is the point where $F(\omega, a_1, a_2, \dots, a_N)$ is required to be maximally flat.

Eq. (1.47) contains a number of independent equations which are equal to the number of the parameters $(a_i) \equiv (a_1, a_2, \dots, a_N)$ and the solution of these equations will give the values of (a_i) which make $F(\omega, a_i)$ maximally flat at $\omega=\omega_0$. The application of maximally flat criterion is widely used in filter theory. The filters having 1) maximally flat amplitude and 2) delay time characteristic at $\omega_0=0$, are known as the "Butterworth" Filter and the "Thomson" Filter respectively. It is necessary to point out that the attenuation characteristic of the "Butterworth" Filter gives the best approximation to an ideal amplitude characteristic at $\omega_0=0$, and similarly, the "Thomson" Filter in delay time characteristic. Filters with maximally flat amplitude and delay time characteristics at ω_0 differing from zero have been recently carried out with the aid of high speed computers [4], [5] (1971).

1.4.2 Least mean square error criterion

The least mean square error criterion approximates an ideal characteristic by minimizing the mean square error in a given interval. This can be further explained by considering an ideal characteristic denoted by K (constant) in an interval $(\omega_1 \dots \omega_2)$ to approximate the function $F(\omega, a_i)$.



Fig. (1-3) Dotted line: typical error function of least square error criterion. Black line: the same error function but in an equiripple manner.

The mean square error will be given as:

$$\phi(a_i) = \frac{1}{(\omega_2 - \omega_1)\omega_1} \int_{\omega_1}^{\omega_2} \{F(\omega, a_i) - K\}^2 d\omega, \quad (1.48)$$

where the error is

$$E(\omega, a_i) = F(\omega, a_i) - K, \quad (1.49)$$

and the parameters (a_i) will be calculated just minimizing $\phi(a_i)$ i.e.

$$\frac{\partial \phi(a_i)}{\partial a_i} = 0 \quad i = 1, 2, \dots, N \quad (1.50)$$

The solution of Eq. (1.50) gives the values of (a_i) which approximate $F(\omega, a_i)$ to be the given constant (k) in a least mean square sense. Practically, in a least mean square approximation, the error is an oscillatory function about the given constant and its magnitude augments near any discontinuity $|6|$. The dotted line in Fig. (1-3) shows a typical least mean square error function $E(\omega, a_i)$ of a lowpass filter.

1.4.3 Equal ripple criterion

The error function $E(\omega, a_1)$ of the least mean square error criterion oscillates about zero in an undefined manner, Fig. (1-3). If the absolute maximum value of the error's ripple in the approximation interval is restricted to minimum, this will lead to an equal ripple error function [6]. This type of approximation is called equiripple approximation (often equivalent to the Chebyshev approximation). This shows that the equal ripple criterion is the optimized form of the least mean square error criterion where the absolute maximum error is restricted to a minimum value which is the ripple magnitude in the approximation interval. This is shown in Fig. (1-3) by the black line which is a typical error function of a lowpass filter in an equiripple sense. A general mathematical form for the equiripple criterion is not known, although the approximation of amplitude $|H(p)|$ and delay time $T(\omega)$ in equal ripple sense are already obtained through direct or indirect methods.

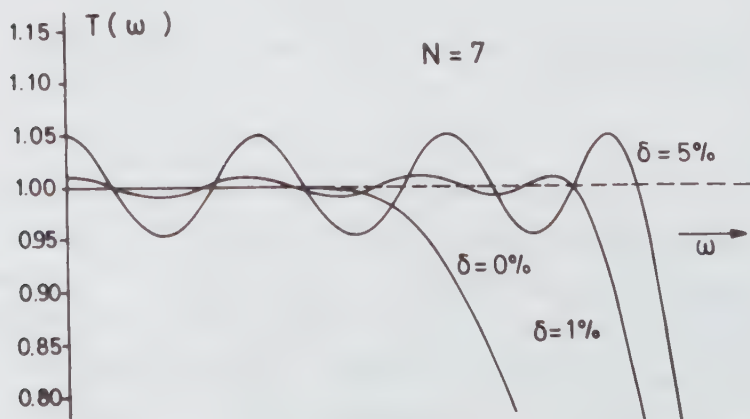


Fig.(1-4) Maximally flat and equiripple approximation of $T(\omega)$.

An important property to be emphasised is that the maximally flat criterion [1.4.1] is a special case of the equal ripple criterion where the ripple magnitude equals zero. Fig. (1.4) shows a maximally flat and equiripple approximation of delay time for different ripple magnitudes.

From the above discussion it is concluded that a useful optimum approximation of a function being constant in a certain interval is its equiripple approximation.

1.5.1 Approximation methods of the phase characteristic

There have been several different attempts to approximate the phase characteristic in an ideal manner. Most work has been done to approximate the delay time to the function

$$T(\omega) = \text{constant} \quad 0 \leq \omega \leq \omega_g \quad . \quad (1.51)$$

Thomson [7] was successful in finding out the polynomial's parameters (a_i) which approximate the delay time in a maximally flat sense. But it was proved later that Thomson's polynomial is only one type of the Bessel polynomial, therefore, the parameters (a_i) are given in a compact form [8] i.e.

$$a_i = \frac{(2N-i)!2^i N!}{(N-i)!i!(2N)!} \quad . \quad (1.52)$$

Later on Ulbrich and Piloty [10] tried to find out the parameters (a_i) which approximate the delay time in an equal ripple sense. They used a complex transformation to put the delay time in a general form of equal ripple rational function [10] and obtained (N) rational nonlinear equations containing (N) unknowns. An iterative scheme is followed to solve these complicated equations assuming uniformly spaced poles on a linear contour as a starting point.

The efficiency and high speed of computers facilitate solving these complicated equations using a special iterative scheme. On the other hand, if the formulation of the problem is changed then the complications may be reduced. Abele [11] has computed the parameters (a_i) which approximate the delay time in an equal ripple sense through a new formulation, which is simpler but none the less still indirect. Although this formulation will be explained more clearly in the next chapter, the following drawbacks will be mentioned here:

- a- contains $(2N+1)$ final equations which are indirect for computation;
- b- these equations are different for even and odd polynomials.

However, it is not as complicated as Ulbrich and Piloty's formulation, the solution of the final equations is carried out with the help of a general computer procedure. It is necessary to mention that the initial values of the parameters are considered to be maximally flat and the initial ripple magnitude is assumed very small. In fact this is a logical selection, because the maximally flat parameters are the solution of the zero equal ripple approximation, sec. (1.4.3).

Another attempt to approximate the phase characteristic is due to Humphreys [12]. In this case the phase function $B(\omega)$ is the approximated function and $(2N)$ equations are formulated directly without transformation. An iterative scheme is used to solve those equations and only some results have been published.

At the end of this chapter a summary of the differences between the three methods of approximation is given in a table. (Tab.1-1). The distinctions between these methods are summarised according

to the following features:

- 1- function which is to be approximated, 2- type of formulation,
- 3- number of equations, 4- parameters, 5- nature of algorithm
- and 6- initial assumed values of the parameters.

Author Characteristic	ULBRICH'S [10]	ABELE'S [11]	HUMPHREYS' [12]
Function to be approximated	$T(\omega)$	$T(\omega)$	$B(\omega)$
Type of formulation	Freq. transf. Iterative transf.	Indirect Without transf.	Direct Iterative transf.
Number of equations	N	$2N+1$	$2N$
Parameters	$H(p)$ -poles	(a_i)	$H(p)$ -poles
Nature of algorithm	Iterative 8-10 cycle	General Procedure	Iterative 5-8 cycle
Initial assumed values of the parameters	Uniformly spaced poles on a linear contour	maximally flat parameters	Uniformly spaced poles on a linear contour

Tab.(1-1) Summary of the differences between the three methods of equiripple approximation.

CHAPTER II (Problem Formulation and Solution)

2.1 Introduction

The approximation of the phase characteristic depends on the approximated function given in (Sec. 1.2.1) and is represented in three forms, namely

$$\begin{aligned} T(\omega) &= T_m & a) \\ V(\omega) &= \text{Zero} & b) \\ B(\omega) &= T_m \omega + B_o & c) \end{aligned} \quad (2.1)$$

where T_m and B_o are constants.

The aim of this chapter is the approximation of the above three forms in an equiripple manner. The problem will be formulated for a suitable computation program to obtain the required solution. The formulation and computation program differ for each of these three forms and to simplify the notations we define:

- Type I: the equiripple approximation of the delay time $T(\omega)$, Eq. (2.1-a),
- Type II: the equiripple approximation of the differentiation of delay time $V(\omega)$, Eq. (2.1-b),
- Type III: the equiripple approximation of the phase function $B(\omega)$, Eq. (2.1-c).

The first part of this chapter is concerned with Type I. Its main properties will be considered together with the formulation for the computation program. The stages of the program and the flow diagram are also given. The second part of this chapter deals with the formulation of Type II. The same procedure is followed as in the first part and, to avoid repetition, the computer program is not given in detail.

In the third part TYPE III is considered. Two cases of Type III namely a) $B_0 = 0$ and b) $B_0 \neq 0$ are differentiated.

The differences between Types I, II and III are emphasised in the analysis and their solutions are given in the form of tables in the appendices. Moreover, in the case where more than one solution exists, examples are given.

2.2 The main properties of the delay time $T(\omega)$ "Type I"

The equiripple approximation of the delay time is worked out in complicated indirect programs |10|, |11|, |12|, already mentioned in chapter I. The resulting complications are due to both their formulation of the problem and the considered properties. In fact, there are some other properties of delay time which will be helpful in solving such approximations with less complication. Considering Fig. (2-1) and the fundamental relations in chapter I, all the important properties of the delay time are summarized as follows

1. $T(\omega)$ is an even function of ω of order $2N$.
2. $T(\omega)$ has a maximum of N maxima and $(N-1)$ minima on the whole axis of ω and, therefore, $(N-1)$ extrema for $\omega > 0$.
3. Every maximum is followed by a minimum and the extreme peaks are maxima.
4. As ω tends to be infinite $T(\omega)$ will tend to be zero i.e.

$$T(\omega) \approx 0. \quad (2.2)$$

5. For a transfer function $H(p)$ given by

$$H(p) = \frac{K}{\sum_{n=0}^N a_n p^n}, \quad (2.3)$$

there are at least $(N-1)$ unknown if the delay time at the origin is given.

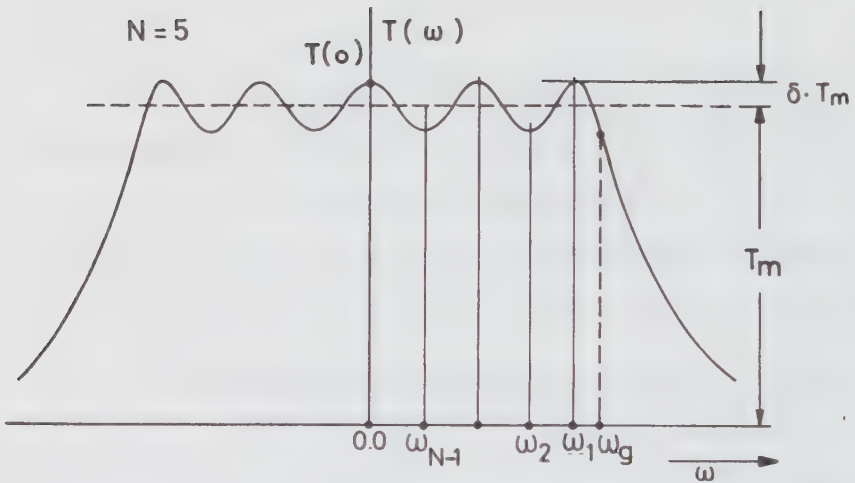


Fig.(2-1) The delay time $T(\omega)$ approximated in an equiripple manner.

To prove this fact the denominator of $H(p)$ is written in real (M) and imaginary (N) terms and then Eq. (2.3) will be

$$H(p) = \frac{K}{M + jN} \quad p = j\omega \quad (2.4)$$

where M , N are the abbreviations of $M(\omega)$ and $N(\omega)$ given in Eq. (1.23). From Eq. (1.25) the delay time will be

$$T(\omega) = \frac{MN' - NM'}{M^2 + N^2} \quad (2.5)$$

where M' , N' are the abbreviations of $\frac{\partial M(\omega)}{\partial \omega}$ and $\frac{\partial N(\omega)}{\partial \omega}$ respectively. The delay time Eq. (2.5) at $\omega = 0$ is

$$T(0) = \frac{N'}{M} = \frac{a_1}{a_0} \quad (2.6)$$

From Eq. (2.3) a_0 can be taken equal to unity without any loss of generality, therefore, Eq. (2.6) will be

$$T(0) = a_1 \quad (2.7)$$

According to Fig. (2-1), the delay time at $\omega = 0$ will be $(1 \pm \delta)T_m$ ($+\delta$ for odd N , $-\delta$ for even N and T_m is the mean delay time), therefore,

$$a_1 = (1 \pm \delta)T_m \quad (2.8)$$

So, in Eq. (2.3) or Eq. (2.5) there are only $(N-1)$ unknowns namely $(a_2, a_3, a_4, \dots, a_N)$, and henceforth they will be denoted as (a_i) .

2.3 Formulation of the Delay Time "Type I"

The formulation of the equiripple approximation of the delay time from Ulbrich and Abele, (Sec. 1.5), and their comparison (Table (1-1)), show that that due to Abele is the simpler mainly because no transformation is required. The formulation followed in the present work is developed on a different basis, which is simpler than both the preceding ones. To show this, the principle formulation of Abele will be first described which makes it possible to compare it with our method, (next subsection).

2.3.1 Abele's Method

Abele has approximated the delay time function $T(\omega)$. The equations necessary to describe the delay time in equiripple form about a constant are not similar in the cases of odd or even N . The procedure to find out these equations is the same for both (odd or even N). The following equations are derived assuming the case of odd N , mean delay time $T_m = 1.0$, and a ripple magnitude of $\pm\delta$. Using Eq. (1.35)

$$T(\omega) = \frac{R(\omega)}{Q(\omega)} \quad (2.9)$$

where $R(\omega)$ and $Q(\omega)$ are even functions of ω , which may be

written as

$$\begin{aligned} R(\omega) &= R(a_j, \omega^{2(i-1)}) \\ Q(\omega) &= Q(a_j, \omega^{2j}) \end{aligned} \quad \begin{aligned} j=0, 1, \dots, N \\ i=1, 2, \dots, N. \end{aligned} \quad (2.10)$$

From Fig. (2-1) considering $T_m = 1.0$ two equations are defined.

The first equation is

$$T(\omega) - (1+\delta) = \frac{R - (1+\delta)Q}{Q}, \quad (2.11)$$

having N real double zeros at $\omega = (0, \pm\omega_1, \pm\omega_3, \dots, \pm\omega_{N-2})$, so that the numerator can be written as

$$R - (1+\delta)Q = K_1 \omega^2 \prod_{\mu=1}^{N-2} (\omega^2 - \omega_\mu^2)^2, \quad (2.12)$$

where K_1 is a constant which is different from zero and $\mu = 1, 3, 5, \dots, N-2$. The second equation is

$$T(\omega) - (1-\delta) = \frac{R - (1-\delta)Q}{Q}, \quad (2.13)$$

having $(N-1)$ real double zeros at $\omega = (\pm\omega_2, \pm\omega_4, \dots, \pm\omega_{N-1})$ and two real simple zeros at $\omega = \pm\omega_g$. The numerator of Eq. (2.16) is

$$R - (1-\delta)Q = K_2 (\omega^2 - \omega_g^2) \prod_{v=2}^{N-1} (\omega^2 - \omega_v^2)^2, \quad (2.14)$$

where K_2 is a constant which is different from zero and $v = 2, 4, \dots, N-1$. If Eq. (2.12) and (2.14) are added and subtracted simultaneously, then

$$\begin{aligned} R - Q &= \frac{1}{2} \{ K_1 \omega^2 \prod_{\mu=1}^{N-2} (\omega^2 - \omega_\mu^2)^2 \\ &\quad + K_2 (\omega^2 - \omega_g^2) \prod_{v=2}^{N-1} (\omega^2 - \omega_v^2)^2 \}, \end{aligned} \quad (2.15)$$

and

$$\begin{aligned} Q &= \frac{1}{2\delta} \{ K_2 (\omega^2 - \omega_g^2) \prod_{v=2}^{N-1} (\omega^2 - \omega_v^2)^2 \\ &\quad + K_1 \omega^2 \prod_{\mu=1}^{N-2} (\omega^2 - \omega_\mu^2)^2 \}. \end{aligned} \quad (2.16)$$

The constants K_1 and K_2 can be determined from the properties of $T(\omega)$ at infinity and zero Eqs. (2.2) and (2.8) respectively. From Eq. (2.10) it can be seen that Eqs. (2.15) and (2.16) are even functions of ω . If the coefficients are set equal on both sides then it will result in $(2N+1)$ independent non-linear equations in $(2N+1)$ unknowns, namely, $(a_0, a_1, \dots, a_N), (\omega_g, \omega_1, \dots, \omega_{N-1})$. These equations are successfully solved through a computer program and the result are published [11].

The maximally flat solution is taken as the starting point for the computer program.

2.3.2 Normalization and General Considerations

Either the delay time or frequency axis can be normalized without any loss of generality. The normalization scale (reference) is arbitrary and is taken for convenience equal to one of the physical quantities in the considered application. The mean delay time T_m will be taken as the delay time normalization scale for lowpass and bandpass approximations. In the case of frequency normalization, either ω_g (the upper edge frequency) or ω_0 (the band center frequency) will be respectively used for lowpass and bandpass approximations.

A) Delay time normalization

Normalizing the delay time $T(\omega)$ with respect to its physical mean value T_m , Eq. (1.26) will be

$$\frac{T(\omega)}{T_m} = \sum_{i=1}^N \frac{-(P_i T_m)}{(T_m \omega)^2 + (P_i T_m)^2} \quad (2.17)$$

Resulting in

- a) The frequency is normalized to $\Omega = T_m \omega$, where Ω is a dimensionless quantity.
- b) The poles of $H(p)$, Eq. (2.3), are multiplied by T_m .

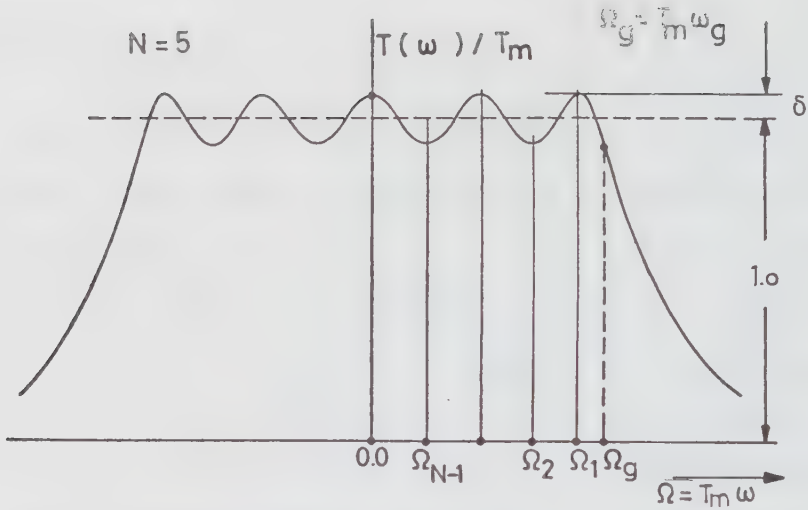


Fig.(2-2) The delay time function normalized to its mean value T_m

This type of normalization is shown in Fig. (2-2).

B) Frequency normalization

Normalizing the frequency with respect to ω_g (the upper edge frequency), Eq. (1.26) will be

$$T(\omega) \cdot \omega_g = \sum_{i=1}^N \frac{-(p_i/\omega_g)}{(\omega/\omega_g)^2 + (p_i/\omega_g)^2} \quad (2.18)$$

This means

- a) The delay time is multiplied by ω_g .
- b) The poles of $H(p)$, Eq. (2.3), are divided by ω_g .
- c) The frequency is normalized to $\Psi = \omega/\omega_g$, where Ψ is a dimensionless quantity.

This type can be seen in Fig. (2-3).

The relation between $\Omega = T_m \omega$ and $\Psi = \omega/\omega_g$ is

$$\Omega = \Psi \cdot T_m \omega_g = \Psi \cdot \Omega_g \quad (2.19)$$

where $\Omega_g = T_m \omega_g$ is the mean delay time-bandwidth product which will be henceforth called the delay-bandwidth product.

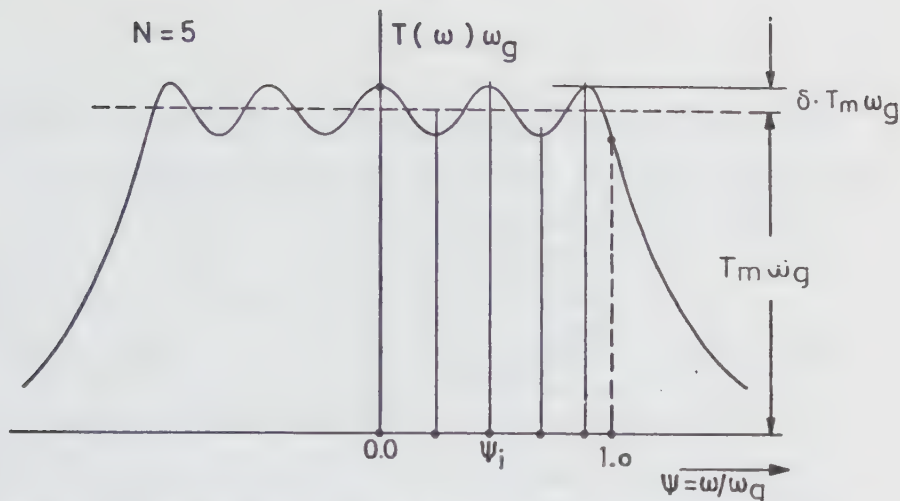


Fig.(2-3) The delay time function when the frequency ω is normalized to the upper edge frequency ω_g .

Eq. (2-19) is basic for the conversion from one type of normalization to the other.

In the following equiripple algorithms the maximally flat parameters are taken as initial values. In the case of the lowpass maximally flat solution, Fig. (1-4), $\omega_g = 0$ which means ω_g can not be used in the frequency normalization for $\delta = 0$. However $T_m \neq 0$ for $\delta \geq 0$, therefore, the delay time normalization with respect to T_m will be applied for the next considered lowpass equiripple approximations. On the other hand, the frequency normalization is convenient to represent the bandpass approximation. The frequency normalization with respect to ω_0 is applied and the results are tabulated. To unify the representation of the results in the appendices, all the solutions are given only for $\delta > 0$ frequency normalized with respect to ω_g or ω_0 for lowpass and bandpass approximations respectively.

2.3.3 The Characteristic Equations

Considering Fig. (2.2), the total number of extrema for $\omega > 0$

is (N-1). Defining a new parameter D as

$$\begin{aligned} D &= +\delta \text{ for a maximum,} \\ D &= -\delta \text{ for a minimum.} \end{aligned} \quad (2.20)$$

At each of the (N-1) (maxima and minima) points $\omega = \Omega_i / T_m$ the following two relations will exist

$$T(\omega) \Big|_{\omega = \Omega_i / T_m} = (1 + D) \cdot T_m, \quad (2.21)$$

$$\text{and} \quad \frac{\partial T(\omega)}{\partial \omega} \Big|_{\omega = \Omega_i / T_m} = 0, \quad (2.22)$$

and if these points (maxima and minima) are called the characteristic points, then Ω_i / T_m will be the position of i-th characteristic point. Substituting $T(\omega) = R(\omega)/Q(\omega)$ and Eq. (1.37) into equation (2.21) then

$$R(\omega) - T_m (1 + D) Q(\omega) \Big|_{\omega = \Omega_i / T_m} = 0 \quad (2.23)$$

$$R'(\omega) - T_m (1 + D) Q'(\omega) \Big|_{\omega = \Omega_i / T_m} = 0, \quad (2.24)$$

where $R'(\omega)$ and $Q'(\omega)$ are respectively the first derivatives of $R(\omega)$ and $Q(\omega)$ w.r.t. ω , and equations (2.23) and (2.24) are the i-th characteristic equations. According to property 2 Sec. (2.2), $T(\omega)$ has (N-1) characteristic points, which means that it has 2(N-1) characteristic equations, for $\omega > 0$. If T_m and δ are given then the number of unknowns in these equations is also 2(N-1), (property 5 sec. 2.2), which consist of (N-1) frequencies (Ω_i) and (N-1) parameters (a_i).

3.3.4 Solution Sequence

From the above normalization discussion, Sec. (2.3.2), either the mean delay time T_m or the upper edge frequency ω_g can be arbitrary choosen whereby the delay-bandwidth product $Q_g = T_m \omega_g$ will be constant for a particular solution.

To simplify and unify all the computation algorithms here, assume T_m or ω_g to be unity, depending on the type of normalization applied, i.e. for the delay time normalization applied in the present analysis, the mean delay time T_m will be considered equal to unity. Now consider the computer algorithm for solving Type I.

Let the maximum at the extreme right hand side, according to Fig. (2.2), lie at $\omega=\Omega_1$ (T_m is unity) and the following minimum at $\omega=\Omega_2$, ... etc.

For a polynomial of order N , the maximum points lie at $\Omega_1, \Omega_3, \dots$ and the minimum at $\Omega_2, \Omega_4, \dots$. This is true in both conditions (odd or even). According to the previous subsection (2.3.3), at each point ($\Omega_1, \Omega_2, \dots$) there are two equations

$$R(\omega) - (1+D)Q(\omega) = Z_1(i) \quad \left| \quad (2.25) \right.$$

$$R'(\omega) - (1+D)Q'(\omega) = Z_2(i) \quad \left| \quad (2.26) \right. \quad \omega=\Omega_i$$

In a computer program it is quite easy to generate^{††} the $2(N-1)$ characteristic equations, Fig. (2-4). Starting at $\omega=\Omega_i$ and $D=\delta$, the first two equations are generated. Repeating the same cycle a further $(N-2)$ times, we obtain $2(N-1)$ equations. Each cycle takes the next characteristic point and changes the sign of D . Fig. (2-4) shows the block diagram to generate the $2(N-1)$ characteristic equations of $T(\omega)$. The solution of these equations depends on:

- the scheme which can solve a set of non-linear equations,
- the initial values of the unknowns.

^{††} Generation of the equations means: Computation of $Z_1(i)$ and $Z_2(i)$ if ω and (a_i) are given.

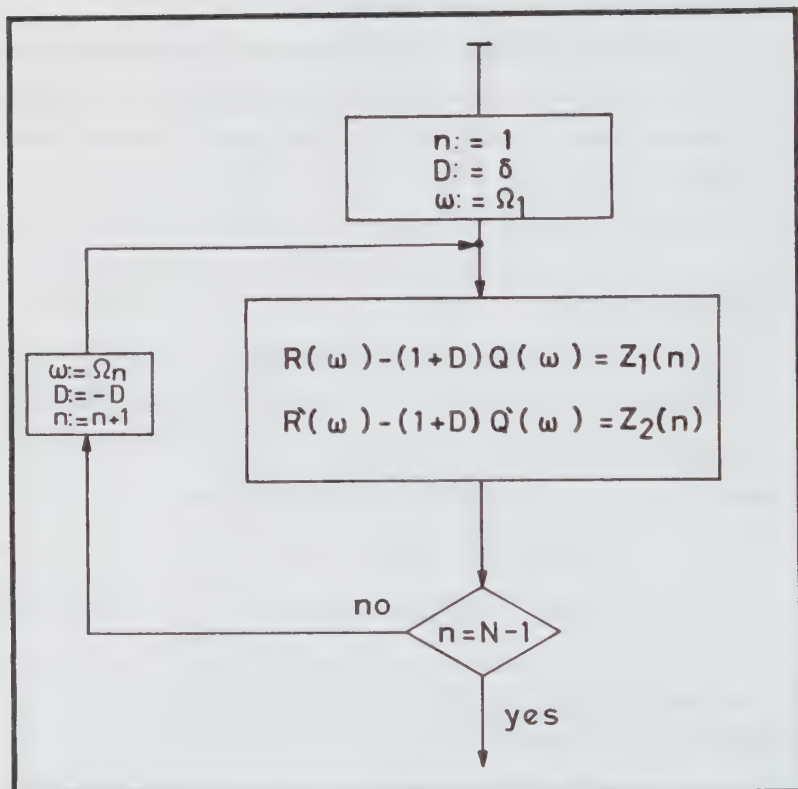


Fig. (2-4) The block diagram for generating $2(N-1)$ characteristic equations.

a) Solving the non-linear equations:

Either a general computer procedure [11] or a special procedure with complicated iterative techniques [10], [12] is used to solve a set of non-linear equations.

The required solution is achieved when $Z_1(i)$, $Z_2(i)$ ($i=1, 2, \dots, N-1$) vanish in Eqs. (2.25), (2.26).

The choice depends upon the formulation of the problem and on the initial values of the unknowns. It is found that a general computer procedure called "GLEI" in the TR 440^{††} library is suitable for solving the equiripple delay time approximation TYPE I using the above mentioned $2(N-1)$ characteristic equations (2.23) and (2.24).

b) The initial values of the unknowns (a_i) and (Ω_i)

The unknowns are divided in two groups:

1. The $(N-1)$ parameters (a_i) where, $i=2, 3, \dots, N$.
2. The $(N-1)$ frequencies (Ω_i) where, $i=1, 2, \dots, (N-1)$.

If the ripple magnitude (δ) is very small, then the maximally flat values of the parameters (a_i) in Eq. (1.52) are considered the most logical initial values of these $(N-1)$ parameters. In the second group, the frequencies (Ω_i) are assumed to be uniformly distributed in a frequency interval equal to $(N-1)$. Therefore, the (Ω_i) initial values may be given by recurrent equations as

$$\begin{aligned}\Omega_1 &= N - c \\ \Omega_{i+1} &= \Omega_i - 1.0\end{aligned}\tag{2.27}$$

where c is a constant which is found to be

$$c=1.0 \quad \text{for } N \leq 6$$

$$c=0.5 \quad \text{for } N=7$$

$$\text{and } c=0.5-(1/N) \text{ for } N \geq 8$$

It is found that the initial values of (a_i) and (Ω_i) are quite far from the final solution, which means that the general procedure has failed to solve these equations. This has led to a procedure, which can refine these initial values being devised and this is shown in Fig. (2-5) and explained as follows:

1. Considering the set (a_i) is temporarily constant and given

by its initial values Eq. (1.52), then the result of solving the

^{††}TR 440 is the name of the Telefunken computer used in Ruhr-Universität Bochum, West Germany.

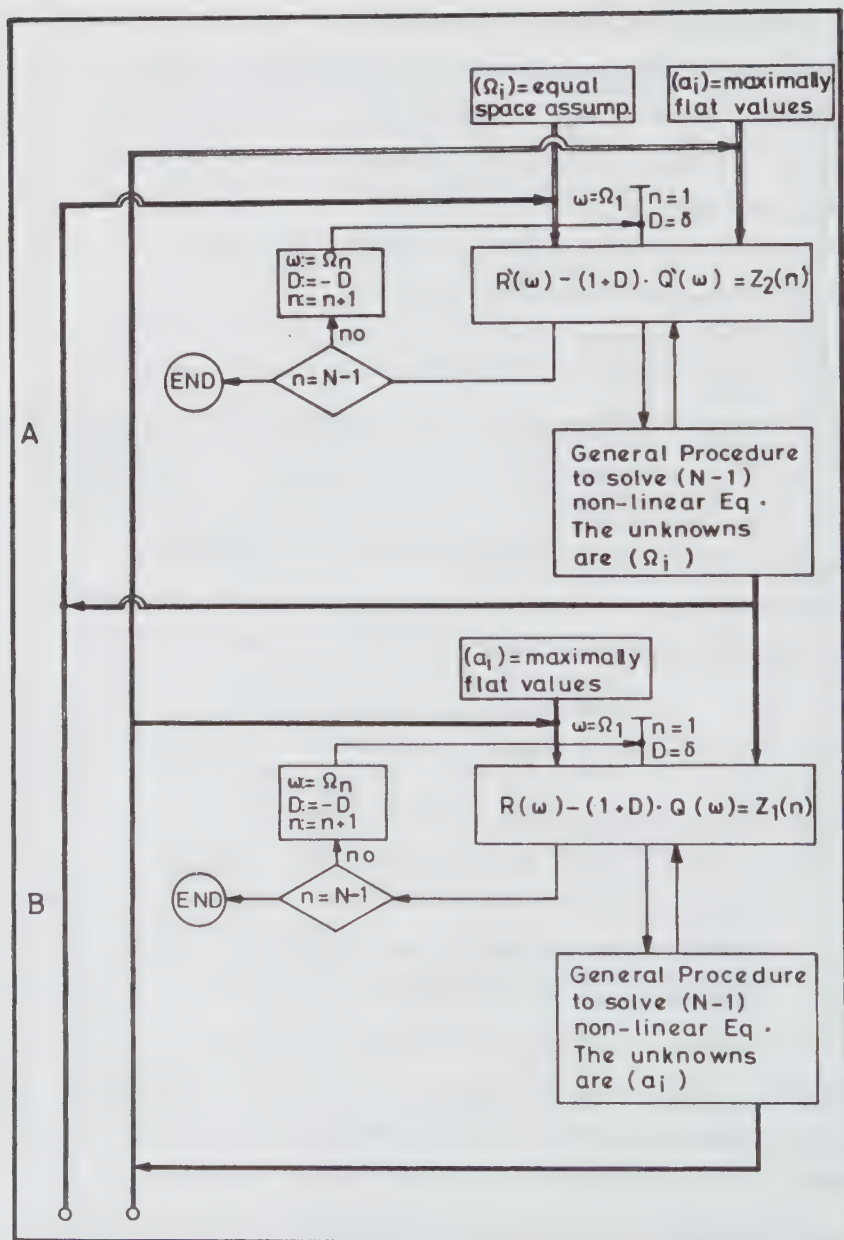


Fig. (2-5) The refining Procedure, part A for the half of the unknowns (Ω_i) and B for (a_i) .

- (N-1) characteristic equations obtained from Eq. (2.26) will be a refined group of the unknowns (Ω_1), (see Fig. (2-5) - A).
2. The resulting values of (Ω_1) are considered to be temporarily constant and the other (N-1) characteristic equations, Eq.(2.25), are used to refine the values (a_1), (Fig. (2-5) - B).
 3. When the above given steps (1 and 2) are repeated, the results (a_1) and (Ω_1) are further improved. In fact, if the number of repetition is increased, the resulting values will converge to the final solution, i.e. $Z_1(i) = 0$, $Z_2(i) = 0$. It is found that the number of repetitions to obtain a very small change (10^{-8}) in parameters is too large. Therefore, in order to save computing time, it is repeated only once, even for large values of N. The resulting values are further used as initial starting values for a general standard procedure to solve the whole 2(N-1) characteristic equations (see Fig. 2-4).
- The complete flow-diagram based on the above method is the com-

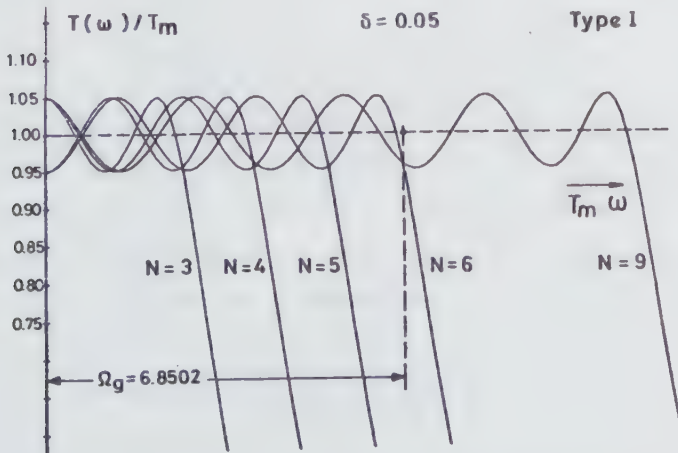


Fig. (2-6)

Equiripple approximation of the delay time $T(\omega)$ for different polynomial orders (N).

combination of the two flow diagrams Fig. (2-4), (2-5) with a general procedure which can solve $2(N-1)$ non-linear equations. The program is quite flexible and it has a large convergence range. This flexibility is best obtained either by increasing the number of cycles in the refining procedure or by decreasing the starting ripple magnitude (δ).

2.3.5 Results

The program is carried out for $N=2, 3, \dots, 9$ and in each case δ is changed from zero percent (maximally flat) to 30 percent of the mean delay time T_m . Fig. (2-6) shows the delay time approximation in equiripple manner for $N=3, 4, 5, 6, 9$ and $\delta=5$ percent. Fig. (2-7) shows the delay time approximation for $N=7$ and different values of ripple magnitude δ .

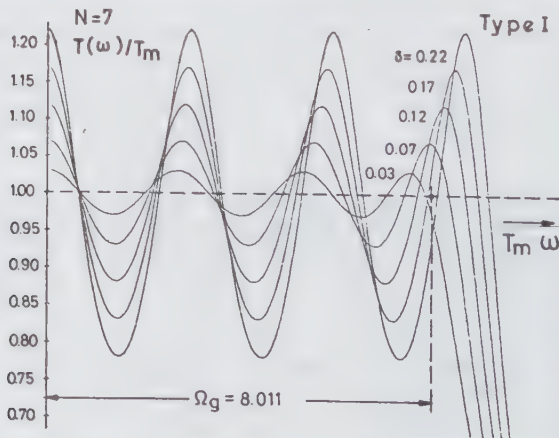


Fig.(2-7) The delay time $T(\omega)$ for different ripple magnitudes.

The values of parameters (a_i) have been published [11] in 6 digits fixed point numbers. Due to the capability of the computer TR 440

(Ruhr-Universität Bochum) the parameters (a_i) are calculated in twelve digits. The frequency normalized poles of $H(p)$ are calculated and given in the form of tables (Appendix III).

These tables show that for all δ 's the real parts of $H(p)$ poles are negative, i.e. the denominator of $H(p)$ is a Hurwitz polynomial.

2.3.6 A possible solution of "TYPE I" with constraint at the origin

In our previous formulation of the problem, the delay time at $\omega = 0$ was not considered as a characteristic point but as a constant value, Eq. (2.8). This constant value defines the approximation interval from zero to ω_g . If the approximation interval does not include $\omega = 0$ then the delay time value at $\omega = 0$ i.e. $T(0)$ is a free parameter.

To demonstrate this, consider equations (2.7) and (2.8), then

$$T(0) = a_1 = (1+D)T_m, \quad (2.28)$$

for the equiripple approximation in the interval from zero to ω_g , and

$$T(0) = a_1 \text{ (free parameter)}, \quad (2.29)$$

for the equiripple approximation in the interval which does not include $\omega = 0$ and extends to ω_g .

It should be noted that the delay time formulation based on the preceeding properties of the characteristic points is quite suitable for solving the constrained delay time $T(\omega, a_1)$ without any change in the problem formulation or in the program stages except giving a_1 externally. The unknown parameters ($a_2 \dots a_N$) are computed and the delay time is plotted for a constant δ and different values of a_1 , Fig. (2-8).

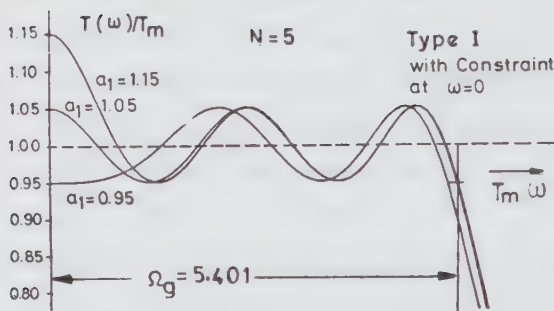


Fig.(2-8) Delay time approximation with constraint at $\omega = 0.0$

This type of constraint is suggested by Rakovich and Djurich (1972) [13] and given without formulation and computer program. It is found [13] that in case of odd-order polynomial the magnitude and unit impulse responses of the resulting network are improved more than in the unconstrained equiripple case.

However, this type of approximation is not an optimal one according to the equiripple approximation (sec. 1.4.3) and there is no criterion to select a_1 , Eq. (2.29), in an optimal manner.

2.4.1 Approximation of the differentiation of delay time $V(\omega)$ in an equiripple manner, "TYPE II"

According to Eq. (2.1), the ideal phase characteristic is achieved keeping the differentiation of the delay time equal to zero i.e.

$$V(\omega) = \frac{\partial T(\omega)}{\partial \omega} = 0. \quad (2.30)$$

Therefore, an equiripple approximation, TYPE II may be carried out by considering $V(\omega)$ instead of $T(\omega)$ as in (Sec. 2.2). The equiripple approximation of $V(\omega)$ is an optimal approximation

about the zero line (Sec. 1.4.3). The differentiation of the delay time $V(\omega)$ can be written in the following compact forms:

a) When the parameters (a_i) of $H(p)$ (Eq. (1.22)) are given then using Eq. (1.42) and Eq. (1.35), one obtains

$$V(\omega) = \frac{MN'' - NM''}{M^2 + N^2} - T(\omega) \frac{2(MM' - NN')}{M^2 + N^2}, \quad (2.31)$$

where M'' and N'' are the abbreviations of $\frac{\partial^2 M(\omega)}{\partial \omega^2}$ and $\frac{\partial^2 N(\omega)}{\partial \omega^2}$

b) When the poles (p_i) of $H(p)$ (Eq. (1.22)) are given, then using Eq. (1.26) of the delay time and differentiating it w.r.t. ω , one obtains:

$$V(\omega) = \sum_{i=1}^N \frac{2\omega p_i}{(\omega^2 + p_i^2)^2}. \quad (2.32)$$

Eqs. (2.31 and 2.32) are helpful in deducing several properties of $V(\omega)$, which are given as follows:

1. Since the poles of $H(p)$ are real or complex-conjugate, $V(\omega)$ is a real function of ω .

2. $V(\omega)$ is an odd function, i.e.:

$$V(\omega) = -V(-\omega) \quad (2.33)$$

$$\text{and} \quad V(0) = 0. \quad (2.34)$$

3. $V(\omega)$ will tend to zero for large ω i.e.

$$V(\infty) = 0. \quad (2.35)$$

4. The equiripple approximation of $T(\omega)$ TYPE I produces a non-equiripple $V(\omega)$ about the ω -axis, (slightly increasing in the magnitude along the positive ω -axis) Fig. (2-9).

5. From Fig. (2-9), for $\omega > 0$, $V(\omega)$ has at most N extrema instead of $(N-1)$ in the case of $T(\omega)$. In the approximation interval $(0 - \omega_g)$ there will be $(N-1)$ extrema whereas one minimum remains outside.

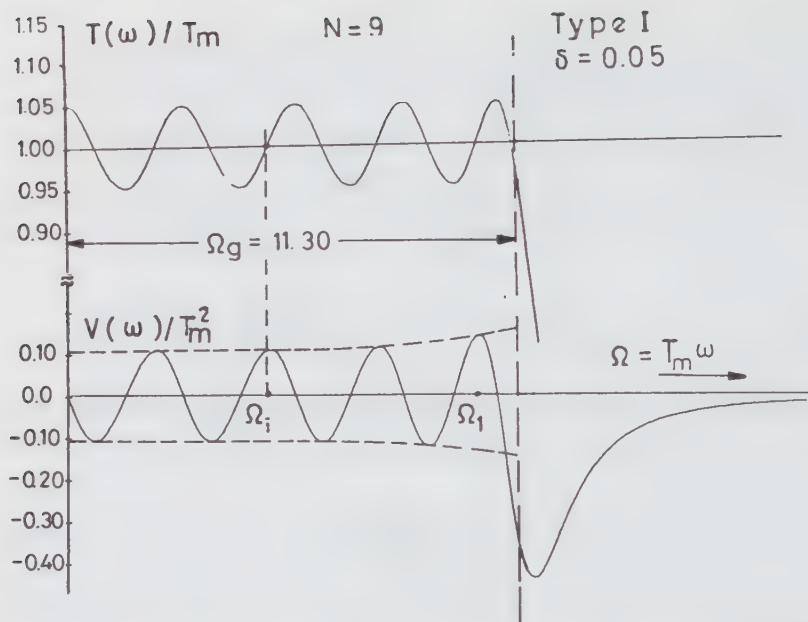


Fig. (2-9) The resulting $V(\omega)$ from Type I approximation.

6. The extrema are alternating i.e. a minimum follows a maximum, and in the approximation interval the extreme right one is a maximum.
7. The area covered by a curve $V(\omega)$ is a constant. This constant is simply calculated by integrating $V(\omega)$ between the limits zero and infinity, i.e.

$$\int_0^{\infty} V(\omega) d\omega = T(\omega) \Big|_0^{\infty}, \quad (2.36)$$

$$= T(\infty) - T(0) = -a_1, \quad (2.37)$$

and the total area between the limits $-\infty$ and $+\infty$ will be equal to zero since

$$\int_{-\infty}^{\infty} V(\omega) d\omega = 0 \quad . \quad (2.38)$$

9. The delay time function $T(\omega)$ and $V(\omega)$ have the same maximally flat parameters, because $V(\omega)$ is the first derivative of $T(\omega)$.
9. The two types of normalization i.e. delay time normalization or frequency normalization can also be considered here. For the delay time normalization, Eq. (2.17), both the frequency and the poles are multiplied by T_m . Considering (2.32) and multiplying both ω and P_i by T_m , then

$$\frac{V(\omega)}{T_m^2} = \sum_{i=1}^N \frac{2(T_m \omega)(P_i T_m)}{\{(T_m \omega)^2 + (P_i T_m)^2\}^2} \quad . \quad (2.39)$$

For the frequency normalization, Eq. (2.18), both the frequency and the poles are divided by ω_g . Considering Eq. (2.32) and dividing both ω and P_i by ω_g , one obtains

$$V(\omega) \cdot \omega_g^2 = \sum_{i=1}^N \frac{2(\omega/\omega_g)(P_i/\omega_g)}{\{(\omega/\omega_g)^2 + (P_i/\omega_g)^2\}^2} \quad . \quad (2.40)$$

This means, normalizing $V(\omega)$ will result in dividing it by T_m^2 in the case of delay time normalization and multiplying it by ω_g^2 in the case of frequency normalization.

10. The $(N-1)$ extrema in the approximation interval in the case of equiripple approximation TYPE II are the characteristic points of $V(\omega)$ i.e. at each point

$$V(\omega) \Big|_{\omega=\Omega_i/T_m} = D \cdot T_m^2 \quad (2.41)$$

$$V'(\omega) \Big|_{\omega=\Omega_i/T_m} = 0 \quad (2.42)$$

($D=\pm\delta$, $+\delta$ for a maximum and $-\delta$ for a minimum) and T_m is the mean delay time. Now substituting Eqs. (1.41) and (1.46) in Eqs. (2.41) and (2.42) will give the characteristic equations, i.e.

$$R'(\omega) - T(\omega) \cdot Q'(\omega) - T_m^2 \cdot D \cdot Q(\omega) \Big|_{\omega=\Omega_i/T_m} = 0 \quad (2.43)$$

$$R''(\omega) - T(\omega) \cdot Q''(\omega) - 2 \cdot T_m^2 \cdot D \cdot Q'(\omega) \Big|_{\omega=\Omega_i/T_m} = 0 \quad (2.44)$$

To obtain the equiripple approximation, the two characteristic equations (2.43) and (2.44) should be realized at $(N-1)$ points (property 5 sec. 2.5).

11. T_m , the mean delay time can be approximately defined from Fig. (2-9) as

$$T_m = T(\omega) \Big|_{\omega=\Omega_i/T_m} \quad \text{for one value } i \text{ in the interval } 1 \leq i \leq N-1. \quad (2.45)$$

Since $T(\omega) = R(\omega)/Q(\omega)$ then

$$R(\omega) - T_m \cdot Q(\omega) \Big|_{\omega=\Omega_i/T_m} = 0 \quad \text{for one value } i \text{ in the interval } 1 \leq i \leq N-1. \quad (2.46)$$

2.4.2 Systematic steps of Solution

Consider the same sequence as in Type I, sec. (2.3.4), for the numerical computation. The delay time normalization is applied whereby T_m is equal to unity. Let the extreme right maximum lie at Ω_1 and the following minimum at Ω_2 , etc. There will be two characteristic equations (2.43) and (2.44) at each point $(\Omega_1, \Omega_2, \dots, \Omega_{N-1})$.

Including Eq. (2.46), the total number of available equations for the approximation of TYPE II will be $2(N-1) + 1 = 2N-1$, where $2(N-1)$ of them are in the form of Eqs. (2.43) and (2.44) and one is in the form of Eq. (2.46). Now, the number of the available equations is equal to the number of unknowns $(a_1, \dots, a_N)$ and $(\Omega_1, \dots, \Omega_{N-1})$. To solve these $(2N-1)$ equations the same procedure is followed as in TYPE I, i.e.

1. Generation of the characteristic equations:

In the generating program Fig. (2-4) Eqs. (2.23 and 24) are replaced by Eqs. (2.42 and 43), considering T_m equals unity.

2. Starting values of (a_i) , (Ω_i) and the refining procedure:

It is possible to adopt the solution of TYPE I for a small ripple magnitude as a starting value of TYPE II. In this case a refining program is not required.

3. General procedure to solve $(2N-1)$ non-linear equations:

The preceding procedure is used namely (GLEI).

The program is arranged according to the above three steps and it is carried out successfully. The values of $(a_1, \dots, a_N)$ are computed given N and δ .

2.4.3 Further analysis of the solution

Eq. (2.46), which is used to define T_m , is not unique. At most T_m can be defined at each of the $(N-1)$ characteristic points i.e. $(N-1)$ definitions. To demonstrate the effect of Eq. (2.46) we consider the following example:

$N = 5$, $\delta = 0.2$ and the value of T_m is set equal to unity. The parameters (a_i) , the characteristic points (Ω_i) and the delay time at each characteristic point are computed. This example is

carried out for $(N-1) = 4$ possible definitions given by Eq. (2.46)
i.e., for $i=1, 2, 3$ and 4. The results are as follows:

$N = 5 \quad \delta = 0.2$

$i = 1$

$a(0, 1, \dots, 5)$

1.0000000000'+000	1.08769070846'+000	4.38964358178'-001
1.37014937580'-001	1.75177326983'-002	2.59441137978'-003

i	Ω_i	$T_n(\Omega_i)^{++}$
-----	------------	----------------------

1	4.60927	1.00000	++
2	3.30494	1.00835	
3	2.10302	1.00654	
4	0.61874	1.00757	

$i = 2$

$a(0, 1, \dots, 5)$

1.0000000000'+000	1.07998456154'+000	4.31287345546'-001
1.33780249347'-001	1.69107191393'-002	2.49355407949'-003

i	Ω_i	$T_n(\Omega_i)$
-----	------------	-----------------

1	4.65633	0.99159	
2	3.33624	1.00000	++
3	2.12467	0.99819	
4	0.62350	0.99923	

$i = 3$

$a(0, 1, \dots, 5)$

1.0000000000'+000	1.08164984419'+000	4.32942077736'-001
1.34475382802'-001	1.70406535926'-002	2.51509896211'-003

i	Ω_i	$T_n(\Omega_i)$
-----	------------	-----------------

1	4.64608	0.99341	
2	3.32943	1.00180	
3	2.11995	1.00000	++
4	0.62246	1.00104	

$i = 4$

$a(0, 1, \dots, 5)$

1.0000000000'+000	1.08069232007'+000	4.31990333615'-001
1.34075426111'-001	1.69658591710'-002	2.50269407918'-003

i	Ω_i	$T_n(\Omega_i)$
-----	------------	-----------------

1	4.65197	0.99237	
2	3.33334	1.00076	
3	2.12266	0.99896	
4	0.62306	1.00000	++

$$^{++}T_n(\Omega_i) = T(\omega)/T_m \Big|_{\omega = \Omega_i/T_m}$$

From the above example it can be noticed that:

1. The values of the parameters (a_i) and the characteristic points (Ω_i) differ slightly in each case.
2. The delay time $T_n(\Omega) = T(\omega)/T_m$ at the characteristic points is not equal to one except at the point where Eq. (2.46) is selected. These are denoted by (++) sign.

As a result of 1. and 2., the definition of T_m Eq. (2.46) leads to different solutions of the equiripple approximation of $V(\omega)$. A possible suggestion to give only one solution may be that the mean delay time T_m will be redefined to equal the average value of $T(\omega)$ throughout the approximation interval instead of at one point as in the preceding case.

T_m is defined as the average value of the delay time at all the characteristic points, i.e.

$$T_m = \frac{1}{N-1} \sum_{i=1}^{N-1} T(\omega) \Big|_{\omega=\Omega_i/T_m} \quad (2.47)$$

Therefore

$$\sum_{i=1}^{N-1} T(\omega) \Big|_{\omega=\Omega_i/T_m} - (N-1) \cdot T_m = 0 \quad (2.48)$$

Replacing Eq. (2.46) by Eq. (2.48) and solving again the previous example, the result is given as

$$N = 5 \quad \delta = 0.2$$

a (0, 1, ..., 5)

1.0000000000'+000	1.08251163969'+000	4.33799333874'-001
1.34835957011'-001	1.71081631270'-002	2.52630250105'-003

i	Ω_i	$T_n(\Omega_i)$
1	4.64079	0.99435
2	3.32592	1.00274
3	2.11752	1.00093
4	0.62193	1.00197
		$T_m = 1.00000$ (average delay time)

As a conclusion, Eq. (2.48) determines that there is only one solution of $V(\omega)$ with a constant average delay time throughout the approximation interval. Therefore, this definition of the mean delay time Eq. (2.48) will be considered from now on.

2.4.4 Results and Characterizing Parameter

The parameters (a_i) are computed for $N = 2, 3, \dots, 7$. The frequency normalized poles of $H(p)$ are calculated for a wide range of δ and given in Appendix IV. In all cases the real parts of the poles are negative i.e. $H(p)$ fulfills the condition that its denominator is Hurwitz polynomial.

The ripple magnitude varies from 0 (maximally flat) to 0.6 which corresponds to a maximum delay-time-ripple of approximately 30 % of the mean delay time. This range is nearly covered in all the cases $N=2, \dots, 7$. Fig. (2.10) shows $V(\omega)$ for different

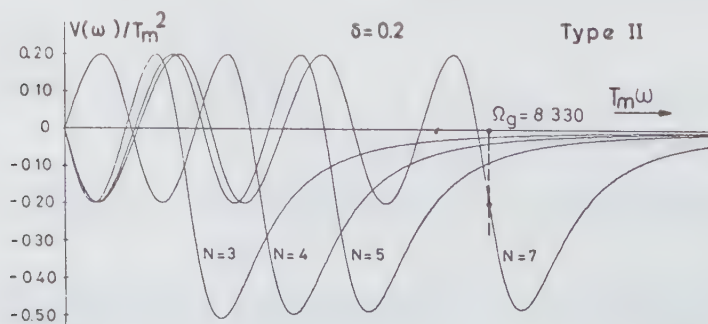


Fig.(2.10) The equiripple approximation of $V(\omega)$ for different polynomial orders(N).

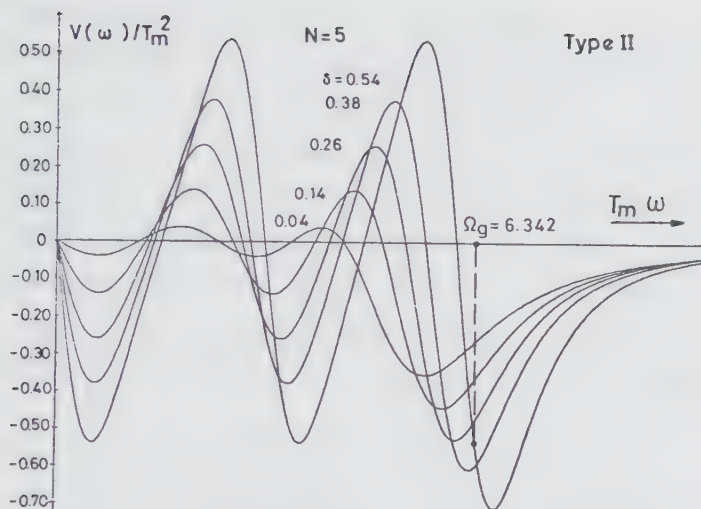


Fig.(2-11) The equiripple approximation of $V(\omega)$ for different ripple magnitudes.

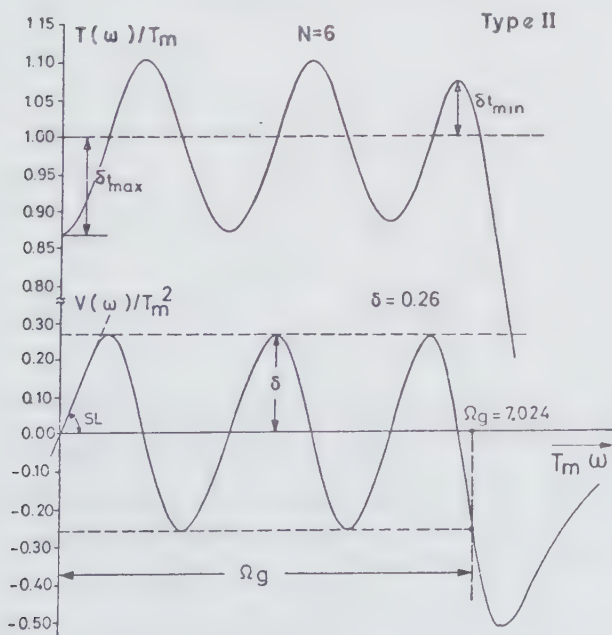


Fig. (2-12) The Parameters which characterize Type II.

values of N and the same δ , while Fig. (2-11) shows the opposite i.e. $V(\omega)$ for the same N and different δ . The delay time $T(\omega)$ resulting from an equiripple $V(\omega)$ is shown in Fig. (2-12). The explanatory remarks for this solution are:

1. The resulting delay time is a ripple function with a ripple magnitude slightly decreasing along the positive frequency direction. The ripples in $T(\omega)$ change from a maximum ($\delta_{\max}$) to a minimum ($\delta_{\min}$) about a constant mean delay time T_m .
2. The parameters which characterize this approximation TYPE II are:
 - a) The ripple magnitude (δ).
 - b) The equal ripple delay-bandwidth product $\Omega_g = T_m \omega_g$.
 - c) The polynomial order N .
 - d) The slope (SL) of $V(\omega)$ at the origin, i.e.

$$SL = \left. \frac{\partial V(\omega)}{\partial \omega} \right|_{\omega=0} . \quad (2.49)$$

SL can be given for an arbitrary N as:

$$SL = 6(a_1 a_2 - a_3) - 2a_1^3 \quad (2.50)$$

The proof of Eq. (2.50) is given in Appendix II.

The role of SL will be clearly seen in the next subsection.

2.4.5 Solution of TYPE II with less characteristic points

It is shown in (sec. 2.3.6), that it is possible to solve the approximation of the delay time $T(\omega)$ TYPE I excluding a_1 . The resulting delay time for a given (N , δ and T_m) is written as $T(\omega, a_1)$.

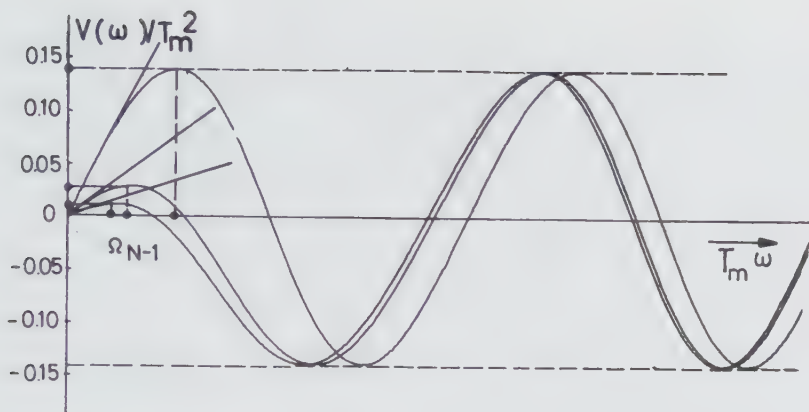


Fig. (2-13) Variation of the slope of $V(\omega)$ at the origin by leaving the $(N-1)$ th characteristic point free

This type of approximation is equiripple in an interval which does not include the origin. The shape of $T(\omega)$ at the origin depends on the external parameter a_1 , see Fig. (2-8).

It will be shown in this section that for TYPE II a solution with an externally selected parameter is also possible. Moreover, the resulting function $V(\omega, \bar{z})$, where $(\bar{z})$ is the external parameter, will have an equiripple form throughout the approximation interval including the origin i.e. for $0 \leq \omega \leq \omega_g$.

From (sec. 2.4.2) $V(\omega)$ has $(2N-1)$ unknowns, namely the parameters (a_1) and (Ω_1) . Each of these parameters can be externally selected as well as a combination of them. It is preferable if the combination of these parameters has a physical meaning. According to the above discussion, the slope of $V(\omega)$ at the origin Eq. (2.50) can be considered as a suitable parameter and consequently the approximation function will be $V(\omega, SL)$.

Fig. (2-13) shows, that keeping N and δ constant, SL can be only varied if the characteristic point next to the origin is left free i.e. its amplitude and position may have any value. The approximation of $V(\omega, SL)$ can be obtained for given N and δ if the preceding computer program of TYPE II is altered in order to:

1. Reduce the number of the characteristic points to $(N-2)$.
2. Solve the equations considering SL as externally given.

The program is carried out successfully and offers the parameters (a_i) for different values of SL and the same N and δ . To show the results consider e.g. $N=6$ and $\delta=0.14$. The initial value of the unknowns is taken directly from Appendix (IV, $N = 6$ and $\delta = 0.14$) and after the conversion to delay time normalization SL is calculated and considered equal to 100 %. Furthermore, the results will be divided into two groups:

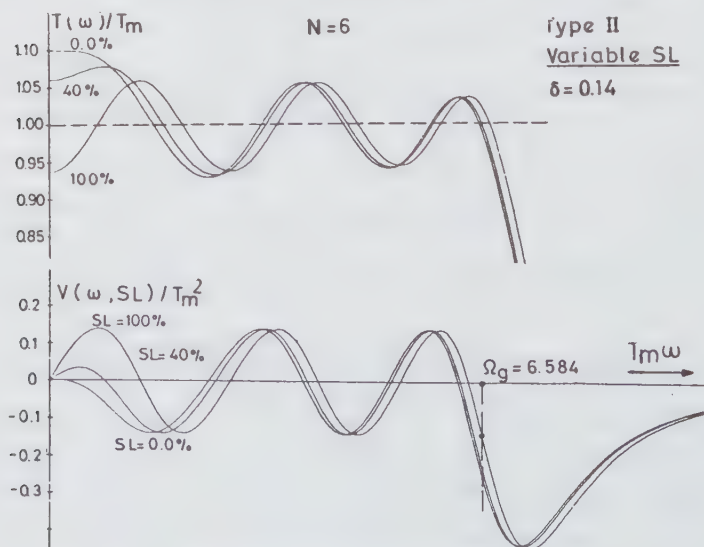


Fig.(2-14) $V(\omega, SL)$ and the resulting $T(\omega)$ for positive values of SL (group (a))

a) for $100 \% \geq SL \geq 0 \%$

b) for $SL < 0 \%$.

A) Fig. (2-14) shows $V(\omega, SL)$ in three cases of group (a) and the resulting $T(\omega)$. The interpretation of these curves is:

1. $V(\omega, SL)$ has $(N-1)$ peaks. The magnitude and position of the $(N-1)$ -th peak (next to the origin) varies as a function of SL . In this case $V(\omega, SL)$ may be considered as an equiripple approximation of $V(\omega)$ with a constraint at the $(N-1)$ -th peak.
2. The delay time function $T(\omega)$ varies about the mean value T_m and its shape at the origin depends on SL .

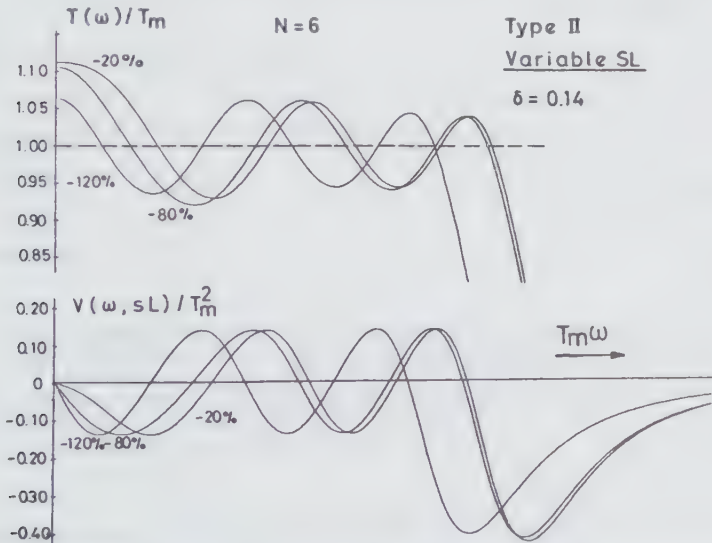


Fig.(2-15) $V(\omega, SL)$ and the resulting $T(\omega)$ for negative SL (group b).

B) Fig. (2-15) shows $V(\omega, SL)$ and $T(\omega)$ for different cases of group (b). We note that:

1. $V(\omega, SL)$ has $(N-2)$ peaks with the same magnitude in the approximation interval. It is an equiripple approximation of TYPE II with one characteristic point less. It is important to mention

here that the equiripple approximation can be carried out for arbitrary number of characteristic points (less than $N-1$).

2. If SL becomes more negative, a_6 vanishes. At this point, where $a_6 = 0$, the polynomial order is automatically reduced by one i.e. $N = 5$. The polynomial parameters (a_i) in this case will be

$$N = 6 \quad \delta = 0.14 \quad SL = -127.52 \%$$

$a(0, 1, \dots, 6)$

1.0000000000'+000	1.05795028823'+000	4.40365760067'-001
1.31490952834'-001	1.76194954906'-002	2.40325830190'-003
1.9472228226'-012		

If there is a unique solution of TYPE II, then the above results, will be the same as in Appendix IV (TYPE II, $N=5$, $\delta=0.14$). In this case the delay time normalized parameters (a_i) are calculated and given as:

$$N = 5, \delta = 0.14$$

$a(0, 1, \dots, 5)$

1.0000000000'+000	1.05795028817'+000	4.40365759958'-001
1.31490952797'-001	1.76194954440'-002	2.40325829736'-003

Considering both results, the values of (a_i) are similar, which, in other words, is a strong indication of the uniqueness of the equiripple approximation.

3. In fact $V(\omega, SL)$ group (b), is a transient solution of the equiripple approximation between $N = 5$ and $N = 6$.

4. For even more negative SL, the parameters a_6 will become negative which leads to unrealizable network.

$$N = 6, \delta = 0.14 \quad SL = -130 \%$$

$a(0, 1, \dots, 6)$

1.0000000000'+000	1.05673573224'+000	4.36941572287'-001
1.29871045159'-001	1.62460043996'-002	2.24471019138'-003
-5.6160783437'-005		

2.5 Equiripple approximation of the phase function $B(\omega)$ "TYPE III"

In chapter I it was shown that the ideal phase characteristic in an approximation interval, Eq. (1.11), should be a straight line, written as

$$B(\omega) = T_m \cdot \omega + B_0 \quad (2.51)$$

where $T_m^{\dagger\dagger}$ is the slope of $B(\omega)$ in the approximation interval

$$T_m = \frac{\partial B(\omega)}{\partial \omega} = T(\omega) \quad (2.52)$$

B_0 is a constant and in the case of lowpass networks it may be equal to zero, since

$$B(\omega) \Big|_{\omega=0} = 0 \quad (2.53)$$

In this case, B_0 is zero and Eq. (2.51) will be

$$B(\omega) = T_m \cdot \omega \quad (2.54)$$

Considering

$$\phi(\omega) = B(\omega) - T_m \cdot \omega \quad (2.55)$$

then the linear condition of the phase function is obtained by keeping $\phi(\omega)$ in the approximation interval equal to zero i.e.

$$\phi(\omega) = 0 \quad (2.56)$$

In the required interval, $\phi(\omega)$ may be approximated about the ω -axis in an equiripple manner. The equiripple approximation of $\phi(\omega)$ about the ω -axis means an equiripple approximation of $B(\omega)$ about a straight line Eq. (2.54). This type of approximation will be called TYPE III.

^{††}For an ideal phase characteristic, Eq. (2.51), T_m coincides with the mean delay time in the approximation interval.

Compact expressions for $B(\omega)$ and $\phi(\omega)$ are:

1. When the delay time function is given, one obtains

$$B(\omega) = \int_0^{\omega} T(\omega) d\omega \quad (2.57)$$

and

$$\phi(\omega) = B(\omega) - T_m \cdot \omega = \int_0^{\omega} (T(\omega) - T_m) d\omega \quad (2.58)$$

2. When real and imaginary parts M and N of the denominator of $H(p)$ in Eq. (1.22) are given, one obtains

$$B(\omega) = \arctan (N/M) \quad (2.59)$$

and

$$\phi(\omega) = \arctan (N/M) - T_m \cdot \omega \quad (2.60)$$

3. When the poles of $H(p)$ Eq. (1.22) are given, i.e.

$$H(p) = \frac{k}{\prod_{i=1}^N (p - p_i)} \quad (2.61)$$

Then

$$B(\omega) = \sum_{i=1}^N \arg(j\omega - p_i) \quad , \quad (2.62)$$

and if $p_i = R_i + jI_i$

Then

$$B(\omega) = \sum_{i=1}^N \arctan \left(\frac{I_i - \omega}{R_i} \right) \quad , \quad (2.63)$$

and

$$\phi(\omega) = \left\{ \sum_{i=1}^N \arctan \left(\frac{I_i - \omega}{R_i} \right) \right\} - T_m \cdot \omega \quad (2.64)$$

2.5.1 Some properties of the phase function $B(\omega)$ and $\phi(\omega)$

Some important properties of $\phi(\omega)$ are given as follows:

1. $\phi(\omega)$ is an odd function i.e.

$$\phi(\omega) = \phi(-\omega) \quad (2.65)$$

and

$$\phi(0) = 0 \quad (2.56)$$

2. $\phi(\omega)$ tends to infinity $(-\infty)$ when ω is very large, i.e.

$$\phi(\infty) = -\infty \quad (2.67)$$

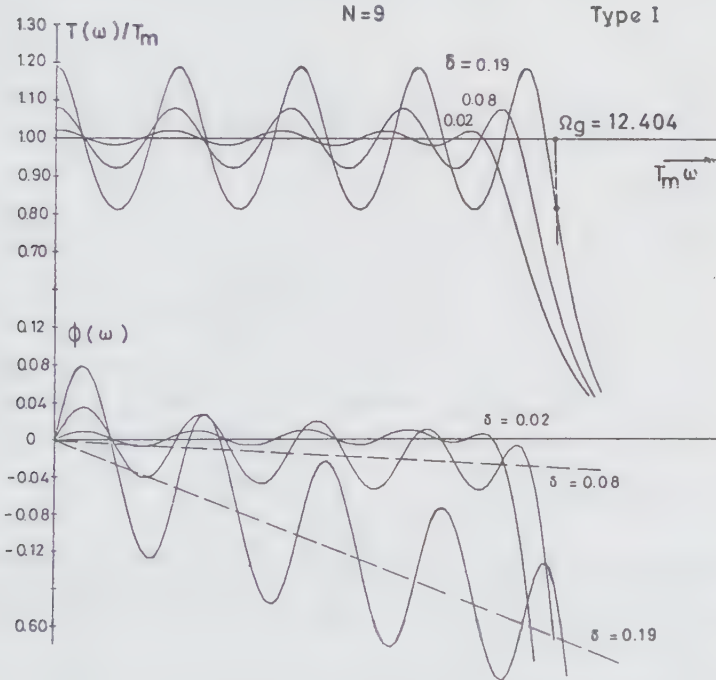


Fig.(2-16) The phase function $\phi(\omega)$ resulting from Type I for different ripple magnitudes.

3. Fig. (2-16) and (2-17) show $\phi(\omega)$ which is resulting from the equiripple approximation respectively of $T(\omega)$ and $V(\omega)$ for different ripple magnitudes. The resulting $\phi(\omega)$ for small ripples is a non-equiripple function about the ω -axis. For large ripples it is also a non-equiripple function about a straight line of negative slope passing through the origin (dotted line).

4. All the extrema of $\phi(\omega)$ are in the approximation interval and their number is at most equal to N . This shows that $\phi(\omega)$ has at

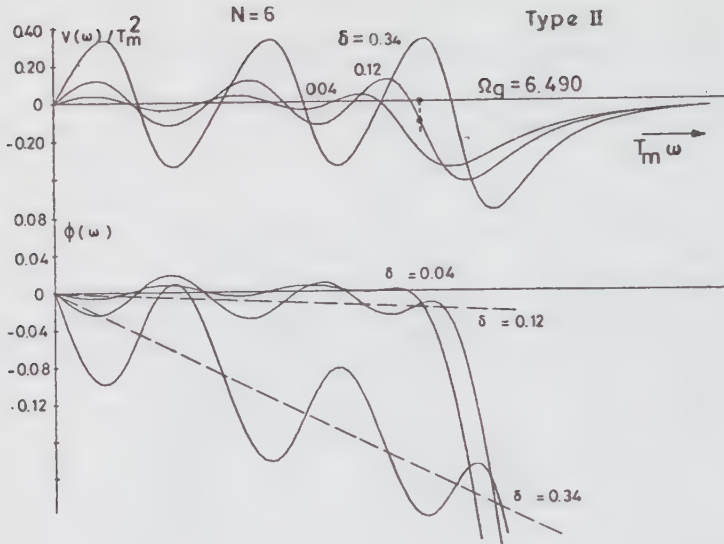


Fig.(2-17) The phase function $\phi(\omega)$ resulting from Type II for different ripple magnitudes.

most N characteristic points.

5. As in the previous sections, for the delay time normalization both the poles and the frequency are multiplied by T_m . The product $T_m \omega$ remains unaffected.

Applying this type of normalization to $\phi(\omega)$, Eq. (2.64) will be

$$\phi(\omega) = \left\{ \sum_{i=1}^N \arctan \frac{I_i T_m - \omega T_m}{R_i T_m} \right\} - T_m \omega \quad (2.68)$$

This means, $\phi(\omega)$ remains the same when applying the delay time normalization.

Applying the frequency normalization, the same result will be obtained. In this case $\phi(\omega)$, Eq. (2.64) will be

$$\phi(\omega) = \left\{ \sum_{i=1}^N \arctan \frac{I_i / \omega_g - \omega / \omega_g}{R_i / \omega_g} \right\} - (T_m \omega_g) \cdot (\omega / \omega_g) \quad (2.69)$$

6. The parameters keeping $\phi(\omega)$ maximally flat will also keep $T(\omega)$ and $V(\omega)$ maximally flat. This is because $T(\omega)$ and $V(\omega)$ are respectively the first and second derivatives of $B(\omega)$.

7. To approximate $\phi(\omega)$ in an equiripple manner, the following two characteristic equations should be valid at each characteristic point

$$\left. \phi(\omega) \right| = D \quad (2.70)$$

$$\text{and} \quad \left. \frac{\partial \phi(\omega)}{\partial \omega} \right| = 0 \quad \omega = \Omega_1 / T_m \quad (2.71)$$

$$\text{However since} \quad \phi(\omega) = B(\omega) - T_m \cdot \omega \quad (2.72)$$

$$\text{then} \quad \left. \frac{\partial \phi(\omega)}{\partial \omega} \right| = T(\omega) - T_m \quad (2.73)$$

Using Eq. (2.73), the characteristic equations will be

$$\left. \phi(\omega) \right| = D \quad (2.74)$$

$$\left. T(\omega) \right| = T_m \quad \omega = \Omega_1 / T_m \quad (2.75)$$

Eq. (2.75) will keep $T(\omega)$ equal to T_m at each characteristic point, i.e. T_m is the mean delay time throughout the approximation interval as in the previous sections.

8. From properties 4 and 7 a set of $2N$ equations are available for $(2N)$ unknowns, namely $(a_1, a_2, \dots, a_N)$, $(\Omega_1, \Omega_2, \dots, \Omega_N)$ and the solution of these $2N$ equations will keep $\phi(\omega)$ in an equiripple manner for given $(N, \delta \text{ and } T_m)$.

2.5.2 Solution Sequence and results

In TYPE III the number of unknowns is equal to the number of available equations, therefore no external definitions of T_m are necessary as in TYPE I and TYPE II, Eq. (2.8) and (2.44)

respectively. A computer program is carried out which differs slightly from preceding ones. These differences are:

1. The equations for the generating program are

$$\arctan(N/M) - \omega = D \quad (2.76)$$

and

$$R(\omega) - Q(\omega) = 0 \quad \omega = \Omega_i \quad (2.77)$$

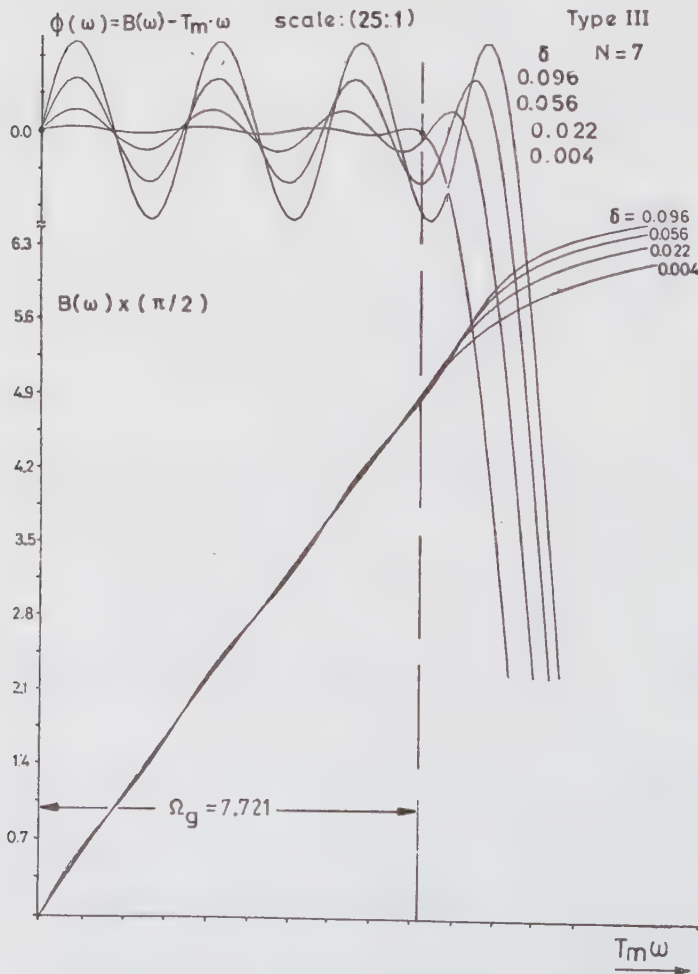


Fig.(2-18) The equiripple approximation of $B(\omega)$ for different values of ripple magnitude.

Eqs. (2.76) and (2.77) are derived directly from Eqs. (2.74) and (2.75), T_m is set equal to unity.

2. The maximally flat parameters, as well as the solutions of TYPE I or TYPE II for small δ may be used as initial parameters to solve TYPE III.

The program is carried out successfully and for different values of (N, δ) the values of the unknowns (a_i) and (Ω_i) are computed. Fig. (2-18) shows $B(\omega)$ and $\phi(\omega)$ for $N=7$ and different values of δ .

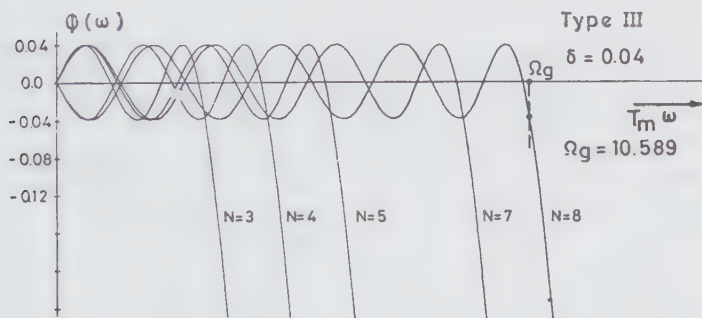


Fig.(2-19) Equiripple approximation of $\phi(\omega)$ for different polynomial orders (N)

Fig. (2-19) shows $\phi(\omega)$ for $N = 3, 4, 5, 7, 8$ and the same ripple magnitude.

It is found that a maximum equal ripple magnitude of 0.1 in $\phi(\omega)$ produce about 30 % maximum ripple in $T(\omega)$ of the mean delay time. In practice this will be the extreme value of the allowable ripple in the delay time. The frequency-normalized poles of $(N=2, \dots, 9)$ and δ varies from 0 to 0.1 are given in appendix V. The real parts of all poles are negative i.e. the denominator of $H(p)$ is a Hurwitz polynomial.

2.5.3 Remarks on the solution of "TYPE III"

1. According to the preceding analysis the number of equations is equal to the number of unknowns. The approximation of ($\phi(\omega) = 0$) is a unique solution, and at all the characteristic points the delay time is equal to T_m Eq. (2.75).
2. The parameters characterizing $B(\omega)$ are selected as in the case of $V(\omega)$ (sec. 2.4.4) namely:

- a), the ripple magnitude (δ),
- b) the equiripple delay-bandwidth product ($\Omega_g = T_m \omega_g$).

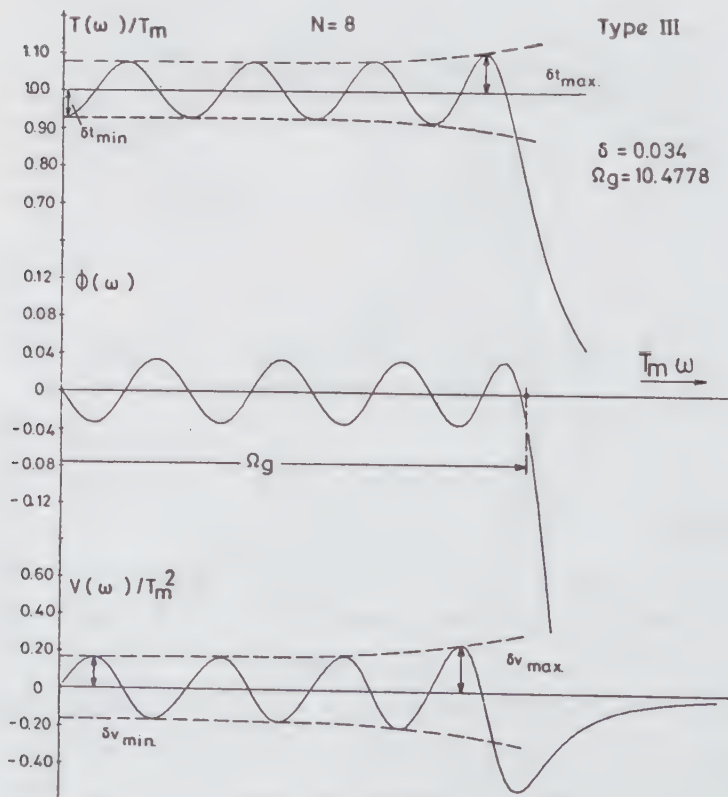


Fig.(2-20) The equiripple approximation of $\phi(\omega)$ Type III and the resulting $T(\omega)$ and $V(\omega)$.

c) The polynomial order (N).

3. The approximation of $\phi(\omega)$ in an equiripple manner, Fig. (2-20), produces slightly increasing ripple in $T(\omega)$ and $V(\omega)$ about the mean value T_m and zero respectively. On the other hand, Fig. (2-16) and (2-17), TYPE I and TYPE II produce $\phi(\omega)$ about a mean line with negative slope.

2.5.4 General solution of "TYPE III"

The approximation TYPE III is carried out by considering B_0 in Eq. (2.51) equal to zero, but in general B_0 can have any value. The main advantage of keeping $B_0 = 0$ is that the approximation interval begins from $\omega=0$. i.e. lowpass range.

In general the ideal phase function Eq. (2.51) is

$$B(\omega) = T_m \cdot \omega + B_0, \quad (2.78)$$

or

$$\phi(\omega) = B_0, \quad (2.79)$$

where $\phi(\omega) = B(\omega) - T_m \cdot \omega$.

Eq. (2.79) is the general form which keeps the phase characteristic in an ideal form throughout the approximation interval.

$\phi(\omega)$ has N characteristic points (property 7 sec. (2.5.1)), and there are two characteristic equations for each namely

$$\phi(\omega) \Big|_{\omega=\Omega_i/T_m} = B_0 + D, \quad (2.80)$$

and

$$\frac{\partial \phi(\omega)}{\partial \omega} \Big|_{\omega=\Omega_i/T_m} = 0. \quad (2.81)$$

Using the same symbols (sec. 2.5.1) the characteristic equations (2.80) and (2.81) will be

$$\arctan(N/M) - T_m \cdot \omega \Big|_{\omega=\Omega_i/T_m} = D + B_0, \quad (2.82)$$

$$R(\omega) - T_m \cdot \omega \Big|_{\omega=\Omega_i/T_m} = 0. \quad (2.83)$$

For the equiripple approximation of $\phi(\omega)$ about B_0 there are $(2N)$ equations i.e. Eq. (2.82) and (2.83) at each characteristic point and $2N$ unknowns $(a_1 \dots a_N)$ and $(\Omega_1 \dots \Omega_N)$. Therefore, for given T_m the parameters (N, δ, B_0) should be given externally to solve the above set of equations. After changing the characteristic equations in the above form, Eq. (2.82) and (2.83), the preceding program (TYPE III) is carried out to solve this set of equations. The initial parameters are taken from the tables (Appendix V TYPE III) and setting $B_0=0$. The program computes the new parameters $(a_i$ and $\Omega_i)$ for different values of N, δ and B_0 .

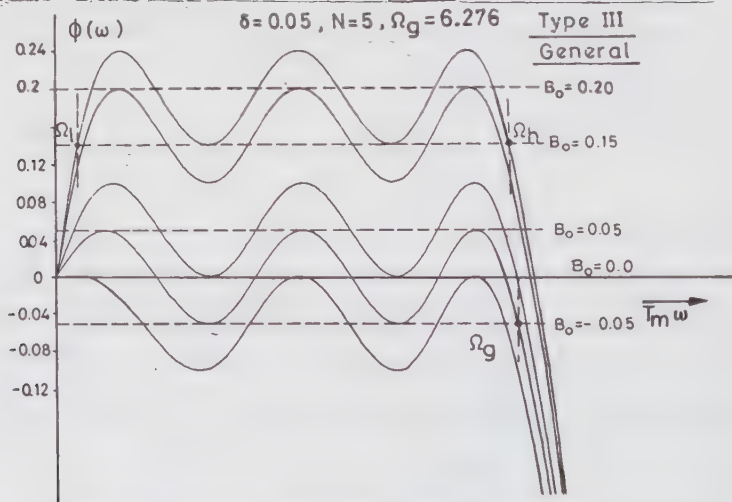


Fig.(2-21) General equiripple approximation of $\phi(\omega)$ for $N=5$ (odd), $\delta=0.05$ and different values of $(B_0 \geq -\delta)$.

To show the characteristics of this general solution of the phase function three diagrams are given namely:

1. $\phi(\omega)$ for $N=5, \delta=0.05$ and five values of B_0 , Fig. (2-21).
2. $\phi(\omega)$ for $N=4, \delta=0.05$ and five values of B_0 , Fig. (2-22).
3. $\phi(\omega)$ for $N=4, B_0=-0.05$ and five values of δ , Fig.(2-23).

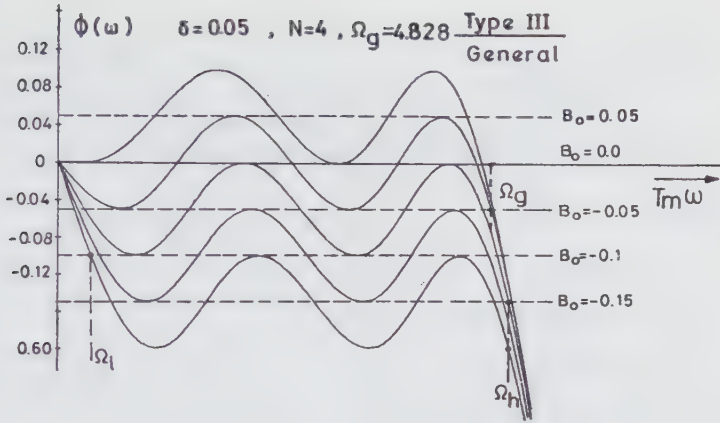


Fig. (2-22) General equiripple approximation of $\phi(\omega)$ for $N=4$ (even), $\delta=0.05$ and different values of $(B_0 \pm \delta)$.

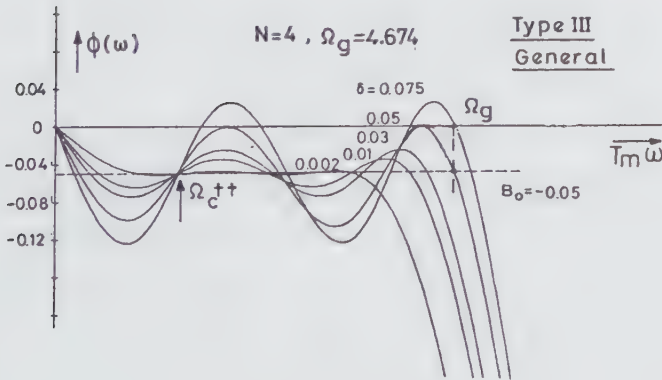


Fig. (2-23) The general Solution of $\phi(\omega)$ for $N=4$, $B_0=-0.05$ and Variable (δ) .

++ For each ripple magnitude (δ) , the crossing point Ω_c is calculated since

$\phi(\Omega_c) = B_0$ and given as :

δ	Ω_c
0.002	1.52543
0.010	1.44450
0.030	1.43171
0.050	1.44099
0.075	1.45489

From these three diagrams the characteristics of the general solution of the phase function may be summarized as:

1. Solutions are only possible in a definite range of B_0 which depends on whether the polynomial order (N) is odd or even. These ranges are:

$$\begin{aligned} \text{a) } B_0 &\leq +\delta \quad (N \text{ even}) \\ \text{b) } B_0 &\geq -\delta \quad (N \text{ odd}) \end{aligned} \quad (2.84)$$

These restrictions, Eq. (2.84), can be deduced theoretically as follows: From property 4 (sec. 2.5), $\phi(\omega)$ has at most N characteristic points. The first one from the right must be a maximum and therefore $\phi(\omega)$ at the N -th characteristic point (next to the origin) will be a) minimum for N even, b) maximum for N odd.

However, a solution of less characteristic points is not considered, then at the N -th point:

$$\begin{aligned} \text{a) } \phi(\omega) &\leq 0 \quad (N \text{ even}) \\ \text{b) } \phi(\omega) &\geq 0 \quad (N \text{ odd}) \end{aligned} \quad (2.85)$$

Substituting Eq. (2.80) in Eq. (2.85) and putting $D = +\delta$ for a maximum (N odd) and $D = -\delta$ for a minimum (N even), we get

$$\begin{aligned} \text{a) } B_0 &\leq +\delta \quad (N \text{ even}) \\ \text{b) } B_0 &\geq -\delta \quad (N \text{ odd}) \end{aligned} \quad (2.86)$$

Eq. (2.86) is correct for any value of (δ) .

2. $\phi(\omega)$ is an equiripple approximation of a constant value over an interval which does not include the origin, i.e. $\Omega_1 \leq T_m \omega \leq \Omega_h$ except for $B_0 = 0$.

3. Fig. (2-24) shows the resulting $T(\omega, B_0)$ and $V(\omega, B_0)$ for two values of B_0 (TYPE III - general). It can easily be seen that $T(\omega, B_0)$ and $V(\omega, B_0)$ are approximately similar to the particular cases which are discussed in (sec. 2.3.6) and (2.4.5), respectively.

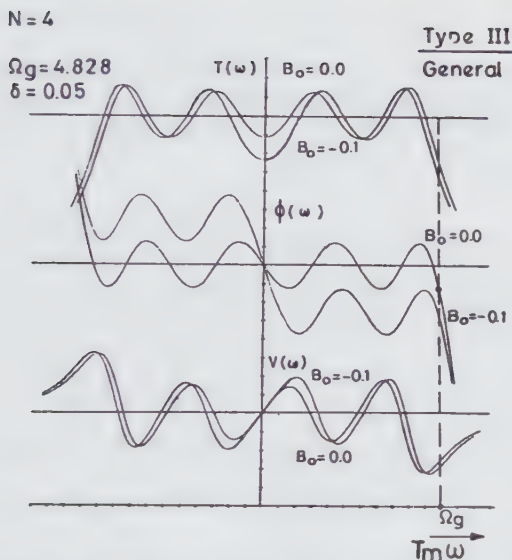


Fig.(2-24) The general approximation of the phase function and the resulting $T(\omega, B_0)$ and $V(\omega, B_0)$.

4. It was mentioned in (sec. 1.4.1) that the maximally flat delay time approximation at $\omega_0 \neq 0$ has been recently worked out [5]. This problem was solved partially through a computer program.

It should be mentioned here that the maximally flat approximation at $\omega_0 \neq 0$ of the phase function (delay time, differentiation of the delay time) can be deduced by solving the general approximation of the phase function (TYPE III-general) for $\delta = 0$ and $B_0 \neq 0$.

Therefore, the maximally flat solution at $\omega_0 = 0$ (Bessel function) is a particular case where $B_0 = 0$ and the general case will be a function of B_0 .

A mathematical expression for this general case may be a good subject for further research.

CHAPTER III

Quantitative Comparison of the Results

3.1 Introduction

In chapter II the parameters of the transfer function $H(p)$ were computed according to three different approximation criteria (TYPE I, II, and III). These approximation criteria are used to approximate the phase characteristic in an optimal manner (equiripple) (sec. 1.4.3).

This chapter is devoted to a detailed comparison between the results so as to select the best one. This will be carried out in both the frequency and time domain.

First a question is put forward, namely:

"How are the results to be compared?". Of course, more than one comparison criterion can be developed having both positive and negative features. It is recommended in the quantitative comparison of the results to use more than one criterion so that the conclusions would be more accurate. Two criteria are developed and used throughout this chapter.

Second, the comparison schemes in frequency and time domains are set out. In the frequency domain the comparison is mainly in:

1. The pole distribution and its variation with the ripple magnitude,
2. The equiripple delay-bandwidth product $\Omega_g = (T_m \omega_g)$ for low-pass networks (only for one of the considered criteria),
3. The attenuation characteristic.

In time domain, the parameters of the unit step response e.g. delay time, rise time, overshoot, etc. are computed and tabulated for different examples. Finally, all the results of the

comparison in frequency and time domains are used to select the best criterion for the equiripple approximation. This comparison scheme will show that TYPE III is throughout the best criterion to be considered in the equiripple phase approximation.

3.2 The Comparison criteria

It is well known that, if $(N, \delta$ and $T_m)$ are given, then the parameters (a_i) of TYPE I, II and III can be computed. Assuming that only N and T_m are given, then (δ) is the only external parameter which defines the set of unknown parameters (a_i) for the specified type. Now if (δ) were selected to be equivalent for all three types, then the computed three sets of parameters would be equivalent.

3.2.1 Average ripple magnitude

Sec. (2.5.2) shows that a constant ripple phase approximation TYPE III results in non-equiripple functions $V(\omega)$ and $T(\omega)$ about constant mean values of 0 and T_m respectively. TYPE I and TYPE II have also produced two non-equiripple phase functions $\phi(\omega)$ about different mean straight lines of negative slope varying as a function of δ (constant ripple magnitude of TYPE I or II) Fig. (2-16,17). Therefore, neither TYPE I nor TYPE II is related to TYPE III in a simple manner since the slope of the mean line of $\phi(\omega)$ is a function of δ . On the other hand, if TYPE III is considered with a constant ripple magnitude δ_c , then the resulting ripple magnitude in $T(\omega)$ about T_m changes from a minimum (δ_{min}) to a maximum (δ_{max}), and similarly for $V(\omega)$ about the ω -axis. Let the average ripple magnitude be given by

$$\delta_{av} = \frac{(\delta_{max} + \delta_{min})}{2} \quad (3.1)$$

Defining δ_{av1} and δ_{av2} as the average ripple magnitude for $T(\omega)$ and $V(\omega)$ respectively, then it is easy to construct unique relations such as $(\delta_c - \delta_{av1})$ and $(\delta_c - \delta_{av2})$ which depend on the polynomial order and the type of approximation used. These relations are computed for different polynomial orders and shown in Fig. (3-1).

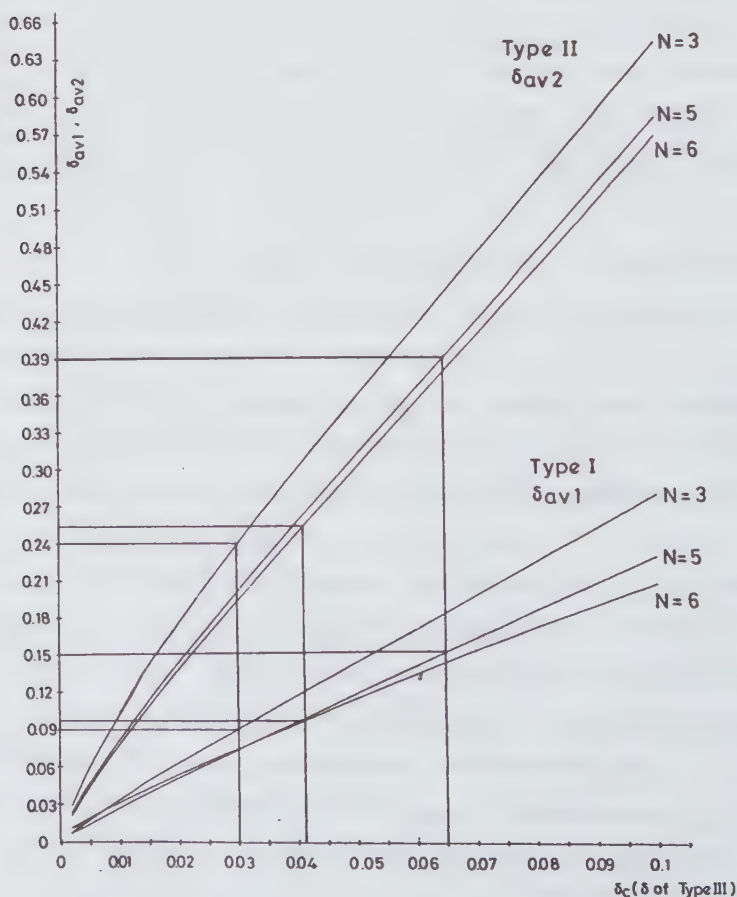


Fig.(3-1) The relation $(\delta_{av} - \delta_c)$ for different polynomial orders(N).

It can be concluded here that

1. The relations are non-linear (especially for small δ_c)
2. They depend on the polynomial order N .

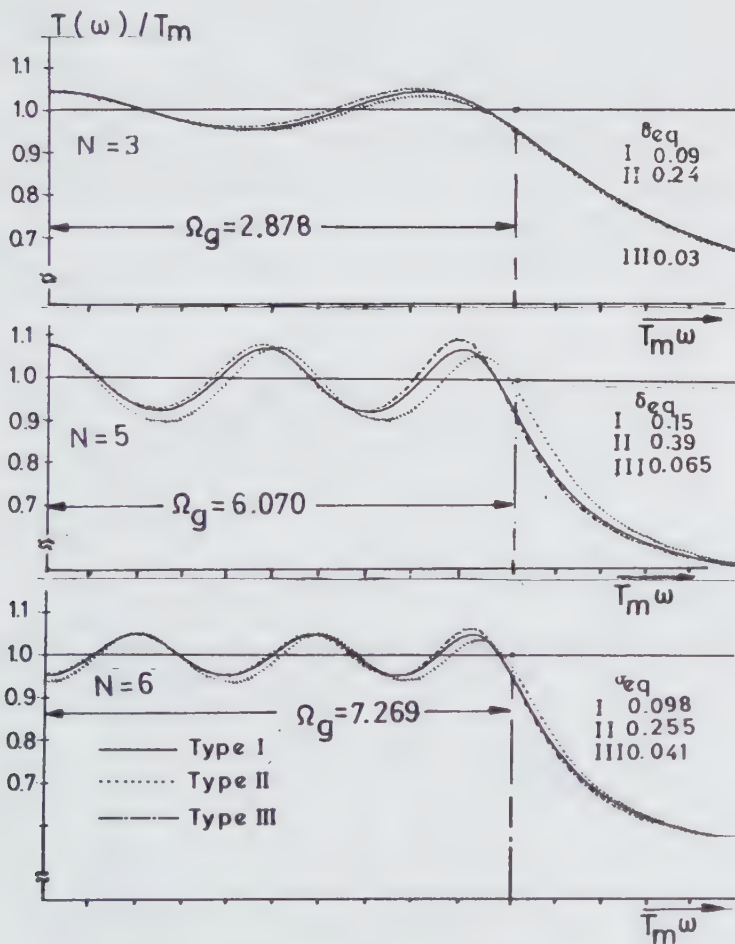


Fig.(3-2) Delay time for three equivalent cases of criterion A for different polynomial orders.

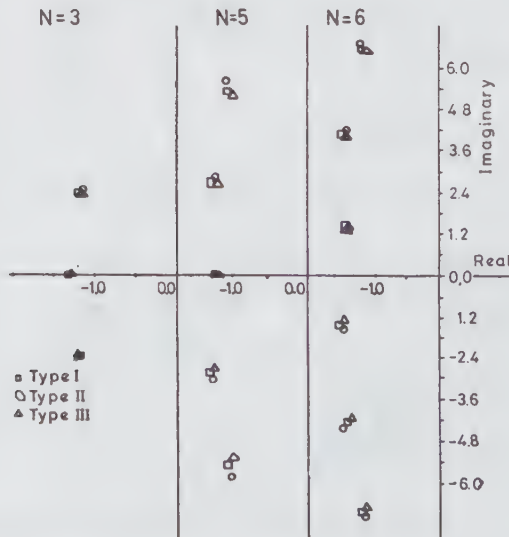
3.2.2 Equivalence Criterion A

For a given δ_c (δ of TYPE III), the corresponding δ_{av1} and δ_{av2} can be found from the curves in Fig. (3-1).

Equiripple criterion A assumes that δ_c and its corresponding δ_{av1} and δ_{av2} are equivalent for TYPES III, I and II respectively.

This is denoted by δ_{eq} for each type.

In each type the set of parameters (a_i) for a given (N, δ) and T_m can either be computed or read out from the tables given in the Appendices. Of course, there is no mathematical proof for the accuracy of the above equivalence but the test in the following section will show that it is quite satisfactory.



Fig(3-3) Poles of three equivalent cases of criterion A for different polynomial orders.

3.2.2.1 Test of the Equivalence Criterion A

To test the previously mentioned criterion:

- a) The delay time function $T(\omega)$ for three equivalent cases is plotted in one diagram for different polynomial orders, Fig.(3-2).
- b) The delay time-normalized poles of the above cases are also plotted Fig. (3-3). From the above plots, Fig. (3-2), and (3-3), it is quite enough to conclude that the equivalence is satisfactory.

3.2.3 Equivalence Criterion B

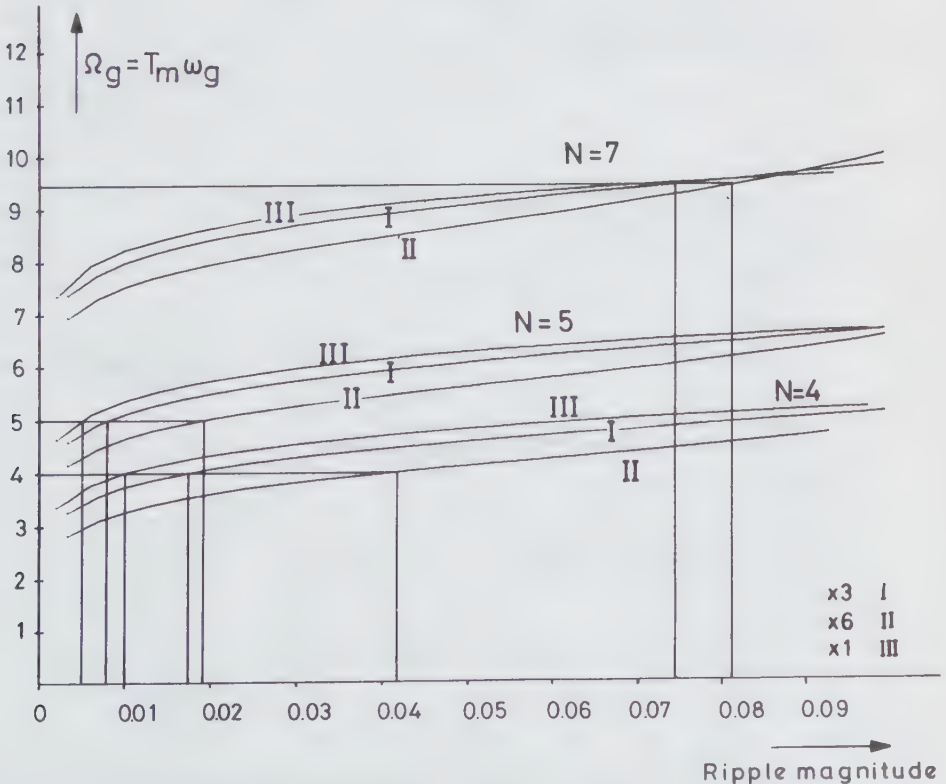


Fig.(3-4) Ripple magnitude - delay time relations for different polynomial orders(N).

In this criterion for the same N two sets of parameters (a_i) in TYPE I, II or III are assumed to be equivalent if they possess the same delay-bandwidth product ($\Omega_g = T_m \omega_g$).

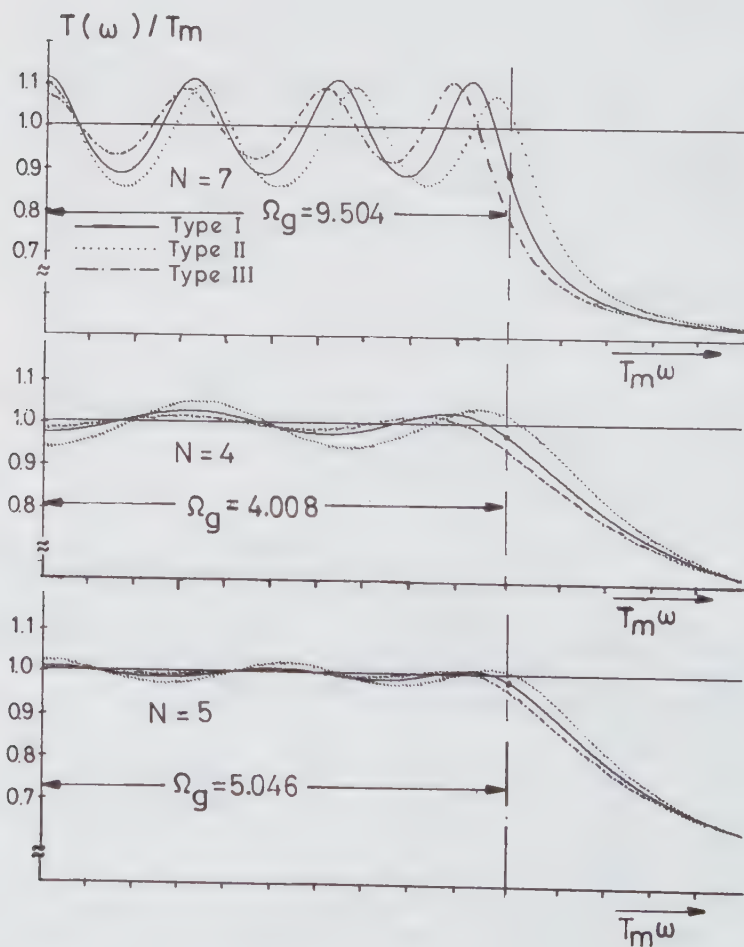


Fig.(3-5) Delay time for three equivalent cases criterion B for different polynomial orders.

In the tables (Appendices III, IV and V), the delay-bandwidth product ($T_m \omega_g$) is given for each set of parameters; the equivalent sets can be selected where the delay-bandwidth products are equal. To be more accurate, the equivalent sets can be found as follows:

- a) Plotting the relation between $\Omega_g = T_m \omega_g$ (ordinate) and δ (abscissa) for the three types in the same figure.
- b) Drawing a horizontal line ($T_m \omega_g = \text{constant}$) which cuts the curves at three different values of δ . These δ 's determine the equivalent sets.

Steps a) and b) can be seen in Fig. (3-4) for different values of polynomial order N .

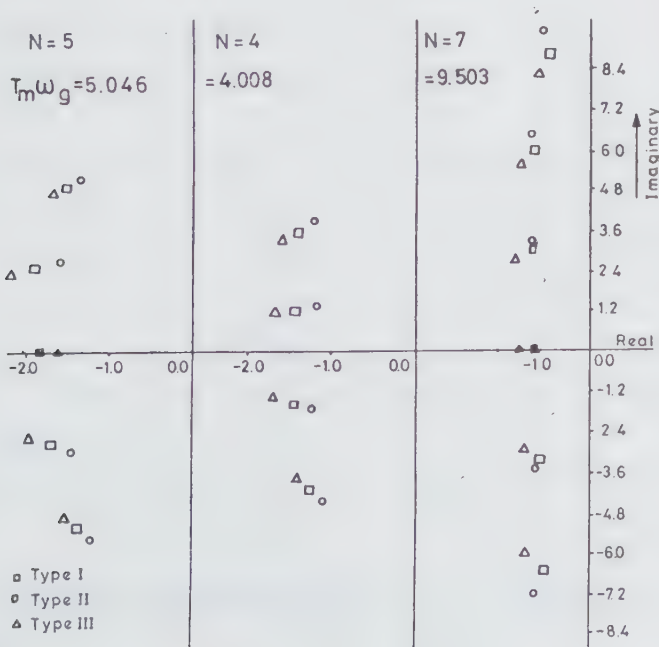


Fig.(3-6) Poles of three equivalent cases of criterion B for different polynomial orders.

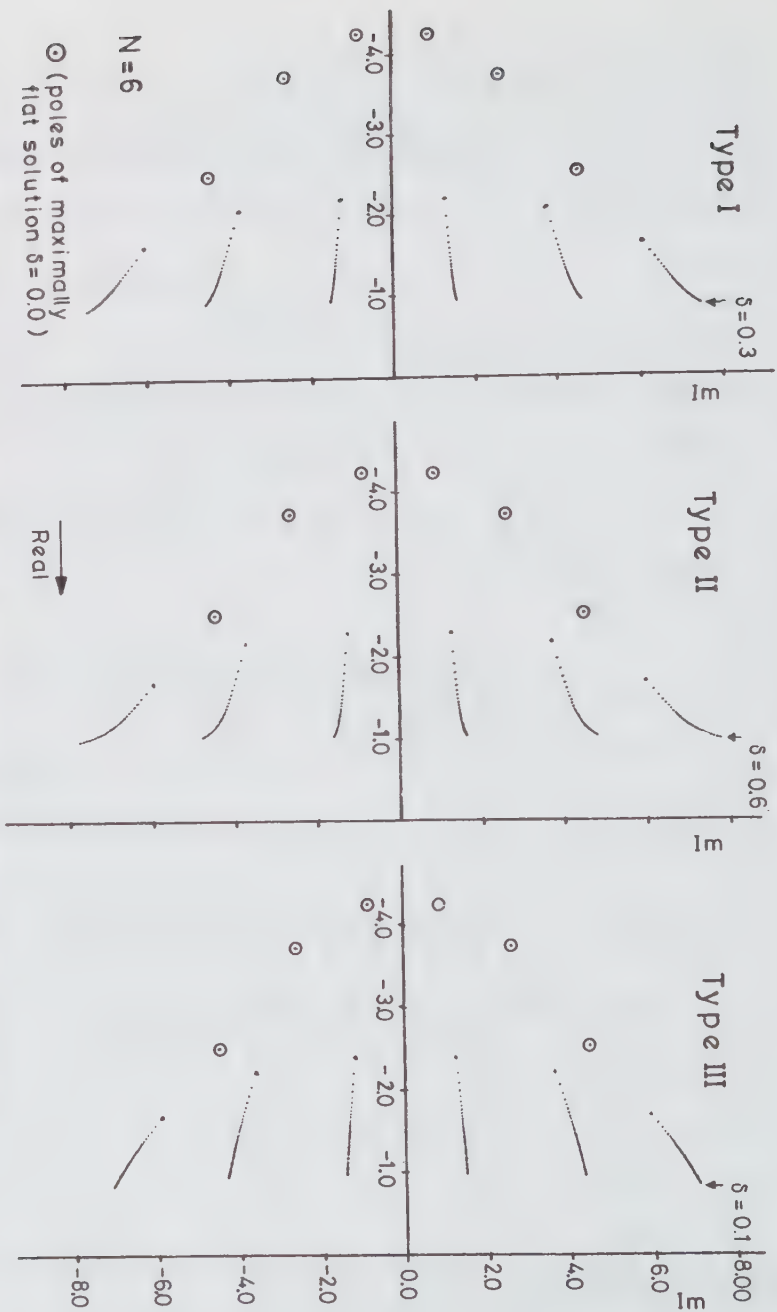


Fig.(3-7) Variation of poles location by varying ripple magnitude for the three types of approximation.

3.2.3.1 Test of the Equivalence Criterion B

Similar curves are plotted as for Criterion A. Fig. (3-5 and 3-6) show respectively the frequency characteristic and the pole distribution of the three equivalent sets of parameters.

From these curves, the equivalence is satisfactory, but not as good as in criterion A. However, the above test is not decisive. Criterion B is the more logical and reasonable base for comparison.

3.3 Variation of poles location by varying the ripple magnitude

An interesting result of the approximation of the phase function is the variation of the poles location by varying the ripple magnitude. The loci of poles (TYPE I, II, and III) are shown in Fig. (3-7) and these are different for each case of approximation.

The interpretation of results in an approximate way is that:

- a) The locus is a straight line for TYPE III.
- b) It is a form of hyperbola for TYPE II.
- c) It is between (1) and (2) for TYPE I.

To measure the deviation of the locus from a straight line in the considered approximation, a minimum-square error straight line is passed through the points and the error is computed.

This can be carried out by considering the pole locus consisting of L points, so the coordinates of the i -th point will be:

$$\begin{aligned}x_i &= \text{Re}(p_i) , \\y_i &= \text{Im}(p_i) ,\end{aligned}\tag{3.2}$$

where p_i is the i -th pole position.

Let the nearest straight line to the locus be

$$y = a x + b .\tag{3.3}$$

The sum of square error measured at each point of the locus will be

$$E = \sum_{i=1}^L \{y_i - (ax_i + b)\}^2 \quad (3.4)$$

The values of the constants a and b which minimize (E) , Eq. (3.4), will define the minimum-square error straight line. The magnitude of the minimum-square error will be sufficient to reflect directly the deviation of the locus of the poles from a straight line relation. This is done for the three types in the range of interest, i.e. for TYPE I ($\delta = 0 \dots, 0.3$) and the equivalent ranges for TYPE II and III. The magnitude of the minimum square error for the three types is depicted in Table (3-1) for $N = 3$.

N = 3	Minimum Square Error
Type I	0.018611503344
Type II	0.143733612982
Type III	0.004020599851

Table (1-3) Minimum square error which reflects the deviation of the locus of a pole ($N=3$) from a Straight line.

From the above table it is clear that for TYPE III the relation between real and imaginary parts of poles can be considered to be a straight line relation in the considered range of the ripple magnitude.

3.4 A comparison between the results in frequency domain

The important frequency response characteristic of a transfer function is the delay-bandwidth product Ω_g . The attenuation characteristic is not considered as a prime factor in our study. Obviously, the comparison of the delay-bandwidth product can only be carried out for criterion A, whereby for the same polynomial order N the function having the largest Ω_g will be considered to be the best.

3.4.1 The delay-bandwidth product

It has been found that $\Omega_g = T_m \omega_g$ varies monotonically with δ in the range of interest. If these relations are plotted for TYPE I, II and III in one diagram where the abscissa is δ_{eq} (computed from the equivalence criterion A, sec. 3.2.2) then it can be concluded from Fig. (3-8) that:

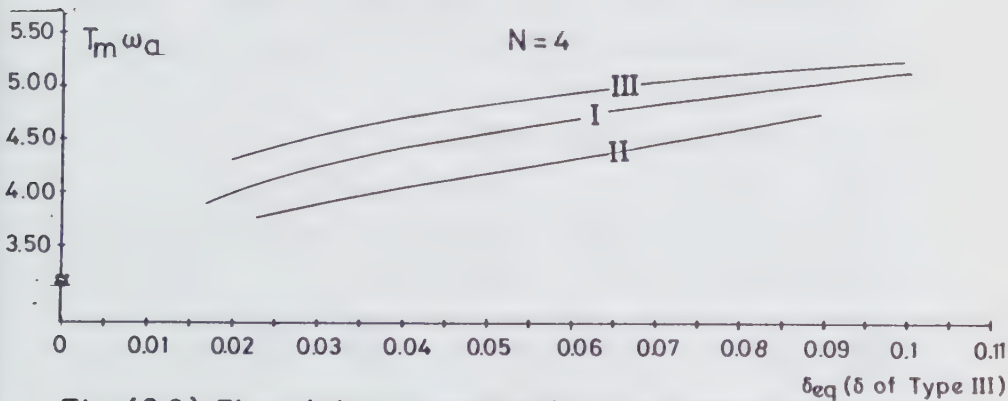


Fig.(3-8) The delay - bandwidth for three equivalent ranges of δ (Criterion A) for Type I,II and III respectively.

All through the equivalence range of ripple magnitude,

$$\Omega_{gIII} > \Omega_{gI} > \Omega_{gII} \quad (3.5)$$

This shows that TYPE III has the largest delay-bandwidth product.

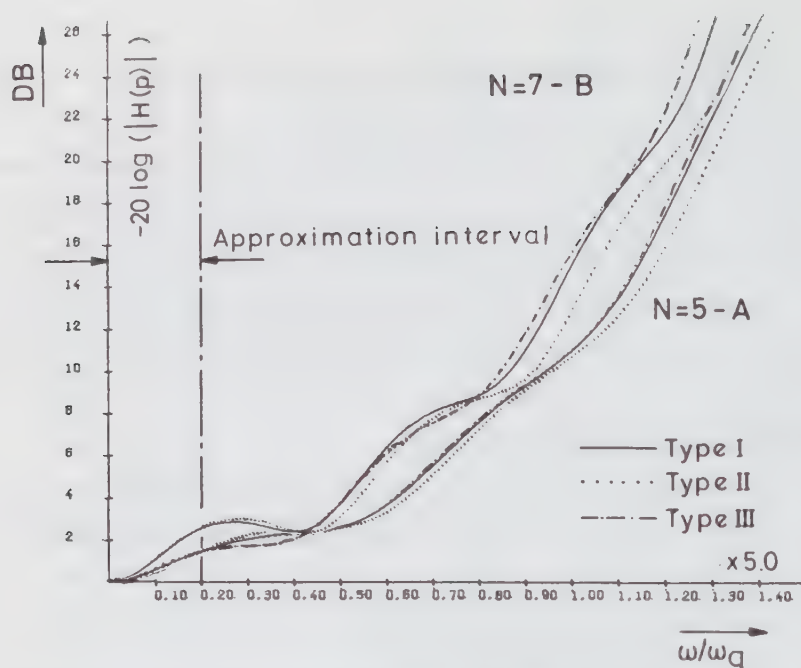


Fig.(3-9) low-pass attenuation of three equivalent cases (criterion A and B).

3.4.2 The attenuation characteristic

The previous study is mainly concerned with the approximation of the phase characteristic. The attenuation characteristic is not a prime parameter for the best phase approximation. Fig. (3-9) shows that the low-pass attenuation characteristic resulting from three equivalent sets of parameters, TYPE I, II and III, according to both criterion A, B. The curves show a small improvement in the case of TYPE III.

3.5 Comparison between the results in time domain

The time reponse characteristics have the same importance as the frequency characteristics. It is common to compare the responses of the various delay-networks under a standard excitation.

The standard excitation is usually either a step function or an impulse function. If $h(t)$ is the impulse response of a network and $G(t)$ is its step response then

$$G(t) = \int_0^t h(t) dt, \quad (3.6)$$

where $h(t)$ is directly given by the poles (p_i) of $H(p)$ in the form:

$$h(t) = \sum_{i=1}^N A_i e^{p_i t}, \quad (3.7)$$

where A_i are real constants.

On the other hand, it is extremely difficult to compare different responses $G(t)$ or $h(t)$ as a whole and so only some extracted quantities from these responses are compared. These are:

1. Delay Time. 2. Rise Time. 3. Overshoot.

There are also other quantities which are not of practical importance. The delay time and rise time are often subject to choice and interpretation [1]. For instance, the delay time has more than five different definitions. Therefore all the quantities used in this comparison program will first be defined.

3.5.1 Definition of the time response quantities

1. 0.5 delay time (T_1):

It is the time lapse until $G(t)$ attains 50 % of its final value.

2. maximum impulse response delay time (T_2):

It is the time lapse until the maximum of the large peak of the impulse response occurs.

3. Nominal delay time (T_D):

This parameter is not so commonly used as the others, but it gives a better description of the delay time [15]. A standard step response of a delay network is defined as:

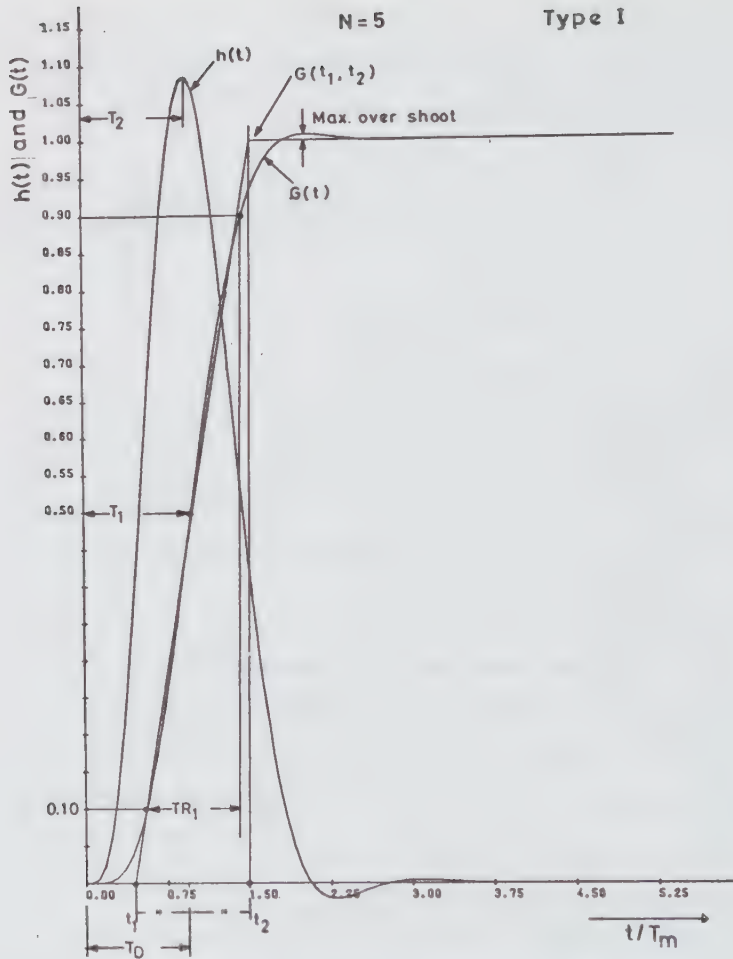


Fig.(3-10) The time domain response and the quantities which characterize it.

The minimum-square error of the three-straight-line segments which pass through the given step response, Fig. (3-10), namely:

- a) The horizontal line from zero to $(t_1, 0.0)$.
- b) The line which connects $(t_1, 0.0)$ and $(t_2, 1.0)$.
- c) The horizontal line from $(t_2, 1.0)$ to infinity.

The total square error between the given step response $G(t)$ and

the three line segments a, b, and c is

$$E = \int_0^{t_1} (G(t))^2 dt + \int_{t_1}^{t_2} \left(G(t) - \frac{t-t_1}{t_2-t_1} \right)^2 dt + \int_{t_2}^{\infty} (G(t)-1)^2 dt. \quad (3.8)$$

If t_1 , t_2 are selected to minimize (E) Eq. (3.8), then the three straight line segments a, b and c will be the standard step response.

The standard delay time is

$$T_D = 0.5(t_2 + t_1). \quad (3.9)$$

For any step response, a standard response in a minimum square-error sense, as given in the above form, can be found. The standard delay time Eq. (3.9) will be called "the nominal delay time".

4. (0.1 -0.9) Rise Time:

This quantity is used more often in practice and is defined as the time lapse for $G(t)$ to rise from 10 % to 90 % of its final value.

5. Küpfmüller Rise Time:

It is the reciprocal of tangent of $G(t)$ at 50 % of its final value

6. Nominal Rise Time:

It is the rise time of standard step response which approximates $G(t)$ in a minimum square sense, considering Fig. (3-10)

$$T_r (\text{nominal}) = (t_2 - t_1) \quad (3.10)$$

7. Minimum Square Error (E):

The magnitude of the minimum square error (E), Eq. (3.8), reflects the deviations of the step response $G(t)$ from the standard one. This error can be considered as a descriptive quantity for the quality of the step response.

8. The normalized error (E/T_D):

It is the ratio of the minimum square error (E), Eq. (3.8), and the nominal delay time (T_D).

9. Overshoot and Undershoot:

This is the maximum amount of excess and recession from the final value of $G(t)$.

3.5.2 Comparison Scheme

The equivalence criterion A and B (sec. 3.2) will be used to select three equivalent sets of parameters (a_i) of TYPE I, II and III. For each of these equivalent cases, the time domain quantities (1, ..., 9), (sec. 3.5.1) are computed and given in the form of tables. Moreover, the step responses are given in a single diagram Fig. (3-12) and (3-13). This procedure is carried out for $N=3,4,6$ and 7. Evaluation and remarks are given under each table.

3.5.3 Results

1. $N = 3$ equivalence criterion A

Type	I	II	III
Ripple magnitude (δ_{eq})	0.09	0.24	0.03
0.5 DELAY TIME	1.0026	0.9995	1.0031
MAX. IMPULSE RESPONSE DELAY TIME	0.9097	0.9021	0.9136
NOMINAL DELAY TIME (TD)	1.0273	1.0258	1.0266
0.1-0.9 RISE TIME	1.1925	1.2025	1.1817
KUPFMULLER RISE TIME	1.2088	1.2122	1.2052
NOMINAL RISE TIME (TR)	1.4320	1.4384	1.4257
RISE-DELAY TIME RATIO (TR/TD)	1.3939	1.4022	1.3869
MINIMUM ERROR (E)	0.0031	0.0034	0.0027
RISE TIME-ERROR PRODUCT (TR.E)	0.0044	0.0049	0.0038
MAX. OVERSHOOT IN PERCENT	-2.283	-2.642	-1.749

Evaluation: TYPE III has the smallest rise time, error, overshoot and the largest delay time.

†† Throughout all the computation, only the delay time-normalized Poles are used.

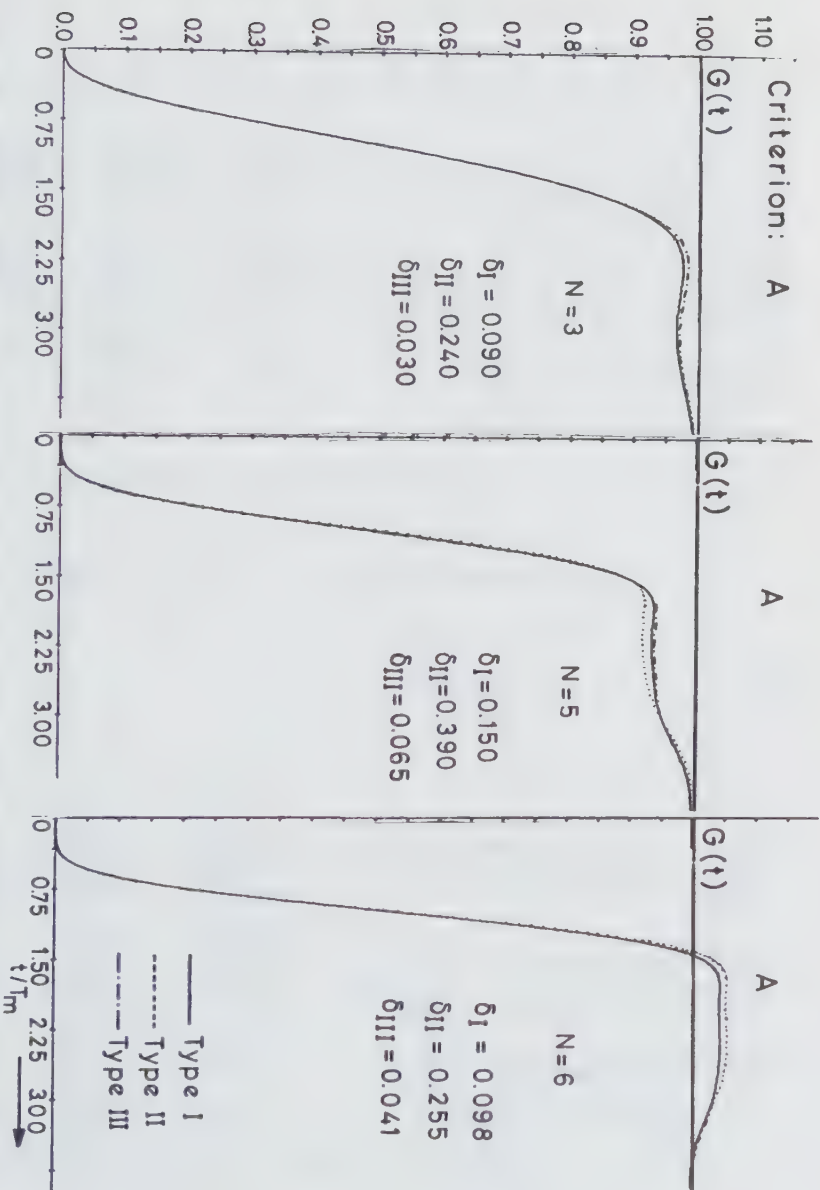


Fig.(3-11) Low-pass unit step response of three equivalent cases (criterion A) for different polynomial orders.

2. N = 5 equivalence criterion A

TYPE	I	II	III
Ripple magnitude (δ_{eq})	0.15	0.39	0.065
0.5 DELAY TIME	1.0174	0.9954	1.0267
MAX. IMPULSE RESPONSE DELAY TIME	0.9800	0.9522	0.9905
NOMINAL DELAY TIME (TD)	1.0278	1.0121	1.0354
0.1-0.9 RISE TIME	0.9114	0.9382	0.9127
KUPFMULLER RISE TIME	0.8624	0.8520	0.8670
NOMINAL RISE TIME (TR)	1.0758	1.0789	1.0783
RISE-DELAY TIME RATIO (TR/TD)	1.0467	1.0660	1.0414
MINIMUM ERROR (E)	0.0080	0.0111	0.0078
RISE TIME-ERROR PRODUCT (TR.E)	0.0086	0.0119	0.0084
MAX. OVERSHOOT IN PERCENT	-6.446	-8.033	-6.259

Evaluation: TYPE III has the smallest error, overshoot and the largest delay time.

TYPE I has the smallest rise time and TYPE III has the smallest rise time-error product.

3. N = 6 equivalence criterion A

TYPE	I	II	III
Ripple magnitude (δ_{eq})	0.098	0.255	0.041
0.5 DELAY TIME	0.9799	0.9656	0.9851
MAX. IMPULSE RESPONSE DELAY TIME	0.9907	0.9794	0.9952
NOMINAL DELAY TIME (TD)	0.9737	0.9598	0.9829
0.1-0.9 RISE TIME	0.6694	0.6531	0.6742
KUPFMULLER RISE TIME	0.6927	0.6781	0.6967
NOMINAL RISE TIME (TR)	0.8130	0.7957	0.8422
RISE-DELAY TIME RATIO (TR/TD)	0.8350	0.8290	0.8569
MINIMUM ERROR (E)	0.0034	0.0046	0.0032
RISE TIME-ERROR PRODUCT (TR.E)	0.0028	0.0037	0.0027
MAX. OVERSHOOT IN PERCENT	4.469	5.411	4.171

Evaluation: TYPE III has the largest delay, the smallest error and overshoot.

TYPE II has the smallest rise time and TYPE III has the smallest rise time-error product.

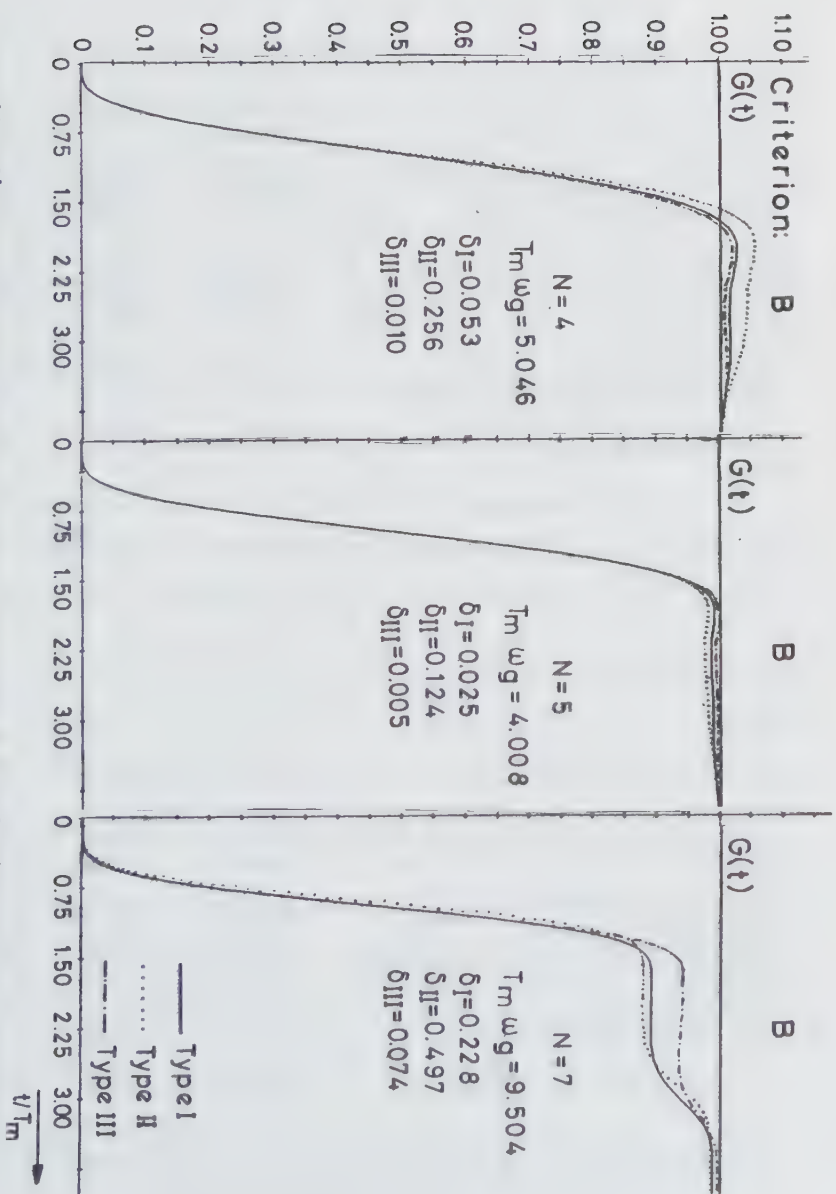


Fig.(3-12) Low-pass unit step response of three equivalent cases (criterion B) for different polynomial orders.

4. $N = 4$ equivalence criterion B $T_m \omega_g = 4.008$

Type	I	II	III
Ripple magnitude (δ_{eq})	0.053	0.256	0.01
0.5 DELAY TIME	0.9830	0.9616	0.9883
MAX. IMPULSE RESPONSE DELAY TIME	0.9624	0.9570	0.9588
NOMINAL DELAY TIME (TD)	0.9875	0.9598	0.9943
0.1-0.9 RISE TIME	0.9016	0.8475	0.9282
KUPFMULLER RISE TIME	0.9404	0.8929	0.9634
NOMINAL RISE TIME (TR)	1.1000	1.0258	1.1216
RISE-DELAY TIME RATIO (TR/TD)	1.1140	1.0688	1.1281
MINIMUM ERROR (E)	0.0015	0.0038	0.0010
RISE TIME-ERROR PRODUCT (TR.E)	0.0016	0.0039	0.0011
MAX. OVERSHOOT IN PERCENT	2.947	5.570	1.901

Evaluation: TYPE III has the largest delay, the smallest error and overshoot.

TYPE II has the smallest rise time and TYPE III has the smallest rise time-error product.

5. $N=5$ equivalence criterion B $T_m \omega_g = 5.046$

Type	I	II	III
Ripple magnitude (δ_{eq})	0.025	0.124	0.005
0.5 DELAY TIME	1.0013	1.0047	0.9990
MAX. IMPULSE RESPONSE DELAY TIME	0.9818	0.9819	0.9800
NOMINAL DELAY TIME (TD)	1.0056	1.0101	1.0031
0.1-0.9 RISE TIME	0.8335	0.8426	0.8328
KUPFMULLER RISE TIME	0.8465	0.8470	0.8493
NOMINAL RISE TIME (TR)	1.0402	1.0456	1.0401
RISE-DELAY TIME RATIO (TR/TD)	1.0344	1.0351	1.0368
MINIMUM ERROR (E)	0.0011	0.0018	0.0009
RISE TIME-ERROR PRODUCT (TR.E)	0.0011	0.0019	0.0009
MAX. OVERSHOOT IN PERCENT	-0.546	-1.800	0.0200

Evaluation: TYPE III has the smallest error and overshoot.

TYPE II has the largest delay time.

TYPE I has the smallest rise time and TYPE III has the smallest rise time-error product.

6. $N = 7$ equivalence criterion B $T_m \omega_g = 9.504$

Type	I	II	III
Ripple magnitude (δ_{eq})	0.228	0.497	0.074
0.5 DELAY TIME	1.0139	0.9653	1.0191
MAX. IMPULSE RESPONSE DELAY TIME	0.9741	0.9236	0.9981
NOMINAL DELAY TIME (TD)	1.0400	0.9999	1.0245
0.1-0.9 RISE TIME	-----	-----	-----
KUPFMULLER RISE TIME	0.7197	0.6916	0.6949
NOMINAL RISE TIME (TR)	0.9785	1.0199	0.8751
RISE-DELAY TIME RATIO (TR/TD)	0.9409	1.0200	0.8542
MINIMUM ERROR (E)	0.0188	0.0218	0.0074
RISE TIME-ERROR PRODUCT (TR.E)	0.0184	0.0222	0.0052
MAX. OVERSHOOT IN PERCENT	-15.90	-17.49	-9.384

Evaluation: TYPE III has the largest delay time, the smallest rise time, error and overshoot.

3.6 The final Evaluation of the Results

Considering the two equivalence criteria A and B, the equiripple approximation of the phase function TYPE III has the most reasonable properties e.g. the greatest delay time, the smallest error and the smallest overshoot. TYPE III has always the smallest error rise time product, but not the smallest rise time. For the equiripple criterion A, the frequency response of TYPE III has the greatest delay-bandwidth product.

It is to mention here that Pfitzenmaier [17], in his study of the form of the delay time for a lowpass transmission function to have optimum time response quantities, has found that:

The lowpass transmission function having delay time of an increasing ripple magnitude in the direction of increasing ω about a constant mean value, will have the best time response.

This result is similar to what we have proved.

This may be clear, if we remember that the magnitude of the delay time ripple resulting in TYPE III is slightly increasing in the direction of increasing ω Fig. (2-20), whereas it is constant or decreasing in the case of TYPE I or II, respectively.

CHAPTER IV

Bandpass Approximation and Filter Implementation

4.1 Introduction

This chapter is devoted to the approximation of the phase function in a bandpass interval. The effect of the standard lowpass-bandpass transformation on the phase, the delay time and the differentiation of the delay time is examined. A review is given for a few trials of bandpass approximation of the phase characteristic. Later on the bandpass approximation with some of its characteristics is shown. The solution and some remarks about the arrangement of tables are given.

The final part of the chapter is a synthesized example of a ladder network possessing lowpass equiripple phase characteristic. This example is tested practically and the results are compiled.

4.2 The effect of the standard transformation on phase characteristic

The standard transformation is given as

$$p = (s+1/s) \quad , \quad (4.1)$$

where p and s are the complex frequencies in the low- and bandpass respectively. This transformation is used to transform a lowpass attenuation characteristic into a bandpass interval, but on the other hand it produces distortion and asymmetry in the phase characteristic [14], [20].

Using the above transformation, the distortion of the phase functions $B(\omega)$, $T(\omega)$ and $V(\omega)$ can be seen from the following:

Consider $p = j\omega$ and $s = j\Lambda$, Eq. (4.1) will be

$$\omega = (\Lambda^2 - 1)/\Lambda \quad (4.2)$$

or

$$\omega = f(\Lambda) \quad , \quad (4.3)$$

where $f(\Lambda)$ is given from Eq. (4.2). The bandpass phase function $B(\Lambda)$ is

$$B_b(\Lambda) = B_1(\omega) \Big|_{\omega=f(\Lambda)} \quad (4.4)$$

$B_1(\omega)$ is a lowpass phase function; In the ideal case it will be

$$B_1(\omega) = T_m \cdot \omega \quad , \quad (4.5)$$

and therefore, Eq. (4.4) will be

$$B_b(\Lambda) = T_m \cdot \Lambda - T_m/\Lambda \quad , \quad (4.6)$$

which shows distortion in the ideal phase characteristic.

In the cases of $T(\omega)$ and $V(\omega)$, the distortion will be greater than in $B(\omega)$. This is explained as follows:

a) From Eq. (4.4), the delay time in the bandpass interval is

$$T_b(\Lambda) = \frac{\partial B_b(\Lambda)}{\partial \Lambda} = \frac{\partial B_1(\omega)}{\partial \omega} \cdot \frac{\partial f(\Lambda)}{\partial \Lambda} \quad (4.7)$$

$$= T_1(\omega) \cdot \frac{\partial f(\Lambda)}{\partial \Lambda} \quad (4.8)$$

b) $V_b(\Lambda)$ is

$$V_b(\Lambda) = \frac{\partial T_b(\Lambda)}{\partial \Lambda} \quad (4.9)$$

Using Eq. (4.8) we get

$$V_b(\Lambda) = V_1(\omega) \cdot \left\{ \frac{\partial f(\Lambda)}{\partial \Lambda} \right\}^2 + T_1(\omega) \cdot \frac{\partial^2 f(\Lambda)}{\partial \Lambda^2} \quad (4.10)$$

The above expressions show that the standard transformation leads to a large distortion in the transformation of the phase functions $B(\omega)$, $T(\omega)$ and $V(\omega)$.

4.3 A Review of the literature for the Bandpass Approximation

Ulbrich and Piloty [10] and Geffe [21] have proposed a method for the transformation of a lowpass delay characteristic of order N to a bandpass characteristic of order $2N$ which shows good delay preserving properties. The transformation is carried out by simply adding (fj) to each scaled-lowpass-frequency-normalized-pole. The scaling factor is $\gamma/2$ where γ is the required relative bandwidth of the bandpass characteristic. The approximate bandpass poles are

$$(\gamma/2) \cdot (p_i/\omega_g) \pm j \quad i=1, 2, \dots, N \quad (4.11)$$

where p_i/ω_g are the lowpass frequency normalized poles.

Ulbrich and Piloty [10] used the above method with a root-improvement-procedure to obtain an exact equiripple delay characteristic in a bandpass interval. Halpern [5] has given a numerical scheme for the maximally flat delay characteristic Eq. (1.47) to preserve it at a finite frequency (bandpass interval).

In fact a certain class of wide-bandpass filter, whose bandpass delay approximate a constant delay about the center frequency, would be worth-while in practice. Blinchikoff and Savetman (1972) [22] have followed the least square error criteria to approximate the wide band constant delay characteristic. The elements of filters of $N = 4$ and $N = 8$ are published. The delay time over the approximation interval oscillates irregularly about the required delay time.

It is well known (sec. 1.4.1) that the ripple magnitude can be equalized and for different ripple magnitudes different solutions will be obtained (Equiripple approximation). In the following, the bandpass equiripple approximation over a relative bandwidth will be analysed.

4.4 Bandpass Approximation

The equiripple approximation of the phase functions are considered in Chapter II and their quantitative comparisons are given in chapter III.

TYPE III namely

$$B(\omega) = T_m \cdot \omega + B_0, \quad (4.12)$$

has the best low pass characteristic in frequency and time domains. Therefore, only Eq. (4.12) in the bandpass approximation will be used. In fact the bandpass approximation of the phase function, Eq. (4.12), is mentioned in the previous approximation of TYPE III-general (Sec. 2.5.4).

For different values of B_0 , the approximation intervals are displaced from the origin. Therefore, the bandpass approximation can be obtained solving TYPE III-general for a suitable value of B_0 .

4.4.1 Approximation for odd and even polynomials

The equiripple approximation for different ripple magnitude (δ) is possible in either the cases of odd or even N . In each case B_0 can be selected only in a certain definite range. Sec. (2.5.4).

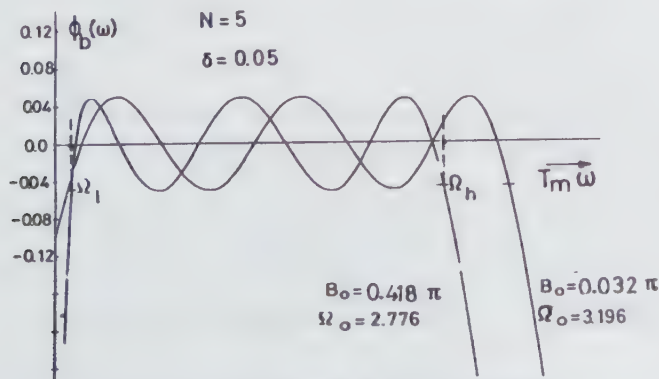


Fig.(4-1) Phase function $\phi_b(\omega)$ (N -odd) for different values of B_0 .

The ranges are

$$\begin{aligned} B_0 &\geq -\delta \quad N - \text{odd} , \\ \text{or} \quad B_0 &\leq +\delta \quad N - \text{even} . \end{aligned} \quad (4.13)$$

Assuming large $|B_0|$, the phase function $\phi(\omega)$, Eq. (2.85) will be shifted up or down and to avoid this we consider:

$$\phi_b(\omega) = B(\omega) - T_m \cdot \omega - B_0 , \quad (4.14)$$

which alternates about the ω -axis with $(\pm\delta)$ (equiripple solution) over the approximation interval.

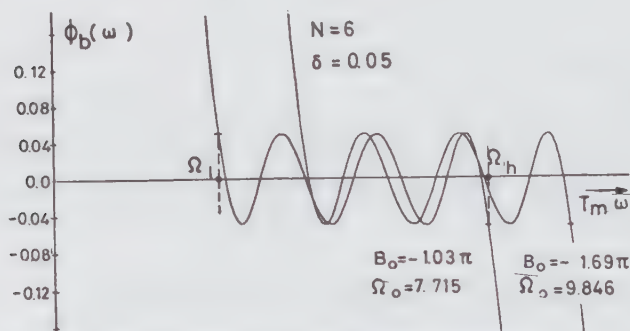


Fig.(4-2) Phase function $\phi_b(\omega)$ (N -even) for different values of B_0 .

Fig. (4-1) and Fig. (4-2) show $\phi_b(\omega)$, Eq. (4.14), for $N = 5$ and $N = 6$ respectively.

The approximation is carried out for two values of B_0 in the given figures. If we denote the upper, and lower edges of the approximation interval as $\Omega_h = T_m \omega_h$ and $\Omega_l = T_m \omega_l$ respectively, then a center value Ω_0 can be defined

$$\begin{aligned} \Omega_0 &= \frac{\Omega_h - \Omega_l}{2} + \Omega_l , \\ \Omega_0 &= \frac{\Omega_h + \Omega_l}{2} = T_m \omega_0 . \end{aligned} \quad (4.15)$$

From the above two figures it is important to notice that for increasing values of B_0 the center value Ω_0 is shifted towards the origin in the case of odd N Fig. (4.1), and it is shifted in the positive direction of ω in the case of even N Fig. (4.2). In other words, for increasing value of B_0 the delay-center frequency product $\Omega_0 = T_m \omega_0$ will become smaller in the case of odd N or greater in the case of even N respectively. This shows that a reasonable bandpass approximation can only be obtained for even ordered polynomial (N -even).

4.4.2 Some properties of the Bandpass Approximation

According to the normalization property of $\phi(\omega)$ sec. (2.5), the ripple magnitude remains unchanged for any type of normalization. If the frequency normalization with respect to ω_0 is considered, then

$$\psi = \omega / \omega_0 \quad (4.16)$$

and the center frequency is

$$\psi_0 = \omega_0 / \omega_0 = 1 \quad (4.17)$$

The relative bandwidth (γ) is defined as:

$$\gamma = \frac{\omega_h - \omega_l}{\omega_0}$$

$$\gamma = \psi_h - \psi_l = (\Omega_h - \Omega_l) / \Omega_0 \quad (4.18)$$

The important parameters of the bandpass phase characteristic are: N (polynomial order), δ (ripple magnitude), γ (relative bandwidth), $\Omega_0 = T_m \omega_0$ (delay-center frequency product) and finally B_0 (constant in Eq.(4.12)).

It is necessary to point out that B_0 has no practical importance for amplitude or frequency modulated signals, (sec. 1.2.2),

i.e. it can be arbitrary chosen, but in the case of unmodulated signals, which are seldom used, the condition for distortionless transmission is

$$B_0 = \pm 2n\pi \quad n=0, 1, 2, \dots \quad (4.19)$$

A bandpass phase approximation under this condition is quite possible but to meet the most practical applications, we consider

$$B_0 = -K\pi \quad (4.20)$$

The relative bandwidth γ is an important parameter in the practical

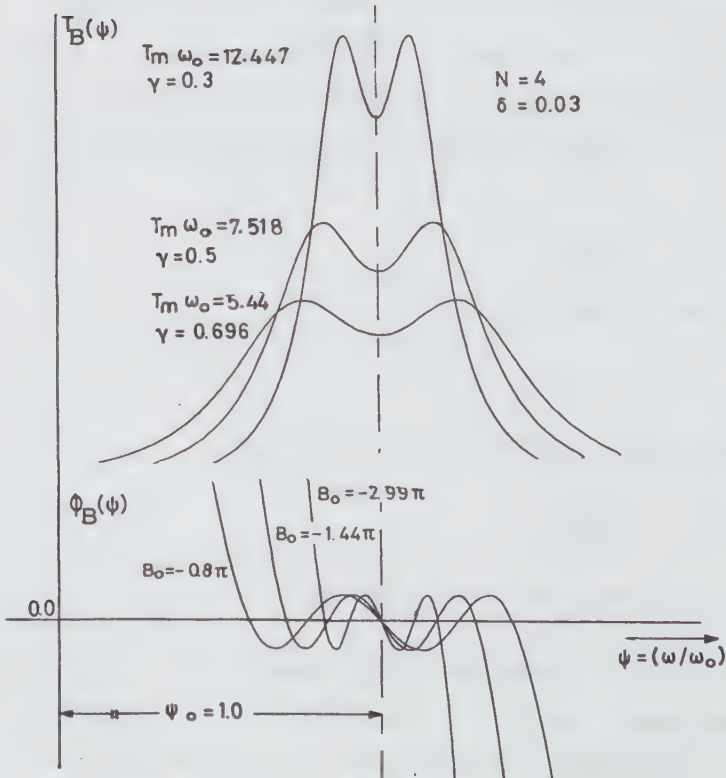


Fig.(4-3) Phase function $\phi_B(\psi)$ and the resulting delay time $T_B(\psi)$ for different relative bandwidths (γ); N and δ are constants.

applications of filters. It can be calculated for a given N , δ and B_0 . In practice, it will be preferable to solve the equiripple approximation for given $(N, \delta$ and $\gamma)$ instead of $(N, \delta$ and $B_0)$. This will be applied in the following section.

The properties of the equiripple approximation of the bandpass phase function may be explained in a reasonable manner through plotting the frequency normalized phase and delay time functions keeping all parameters constants except the one under consideration. Considering a frequency normalization by ω_0 , the phase and delay time function will respectively be

$$\phi_B(\psi) = \phi_b(\omega) \quad (4.21)$$

$$\text{and} \quad T_B(\psi) = T(\omega)\omega_0 \quad (4.22)$$

1. Fig. (4-3) is plotted to show that for constant N and δ , the relative bandwidth (γ) and the delay-center frequency product ($T_m\omega_0$) will vary monotonically with B_0 . It shows $\phi_B(\psi)$ and $T_B(\psi)$ for $N=4$, $\delta=0.03$ and different values of B_0 .

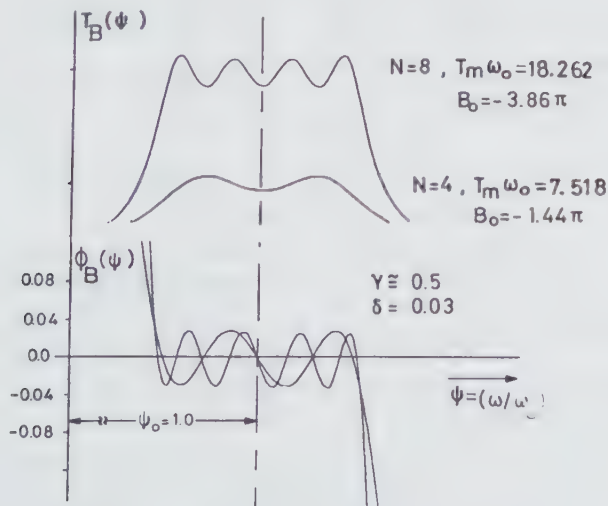


Fig. (4-4) $\phi_B(\psi)$ and the resulting $T_B(\psi)$ for different polynomial order (N); γ and δ are constants.

2. For different polynomial order N , the same relative bandwidth can be achieved for different values of the delay-center frequency product. The delay center frequency product increases for large value of N . This is shown in Fig. (4-4) where $\phi_B(\psi)$ and $T_B(\psi)$ for $N=4$ and $N=8$, ($\delta=0.03$ and $\gamma=0.9$) are plotted.

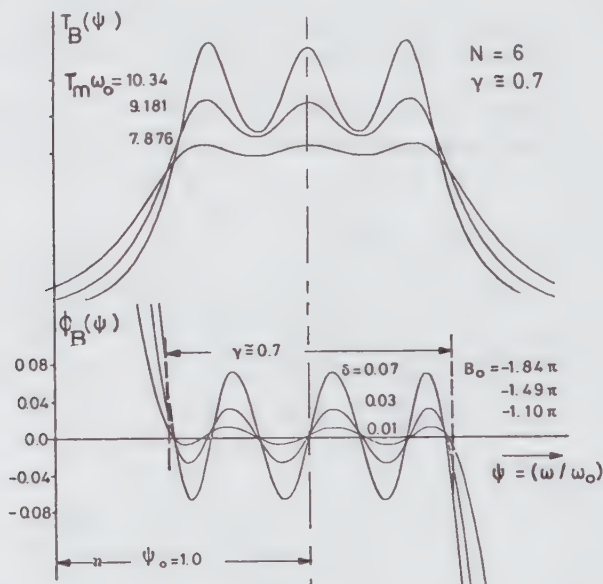


Fig.(4-5) $\phi_B(\psi)$ and $T_B(\psi)$ for different values of ripple magnitude (δ); N and γ are constants.

3. The variation of δ , for the same relative bandwidth γ and polynomial order N , will have a slight effect on the delay-center frequency product ($T_m \omega_0$). This can be seen in Fig. (4-5) for $N=6$, $\gamma=0.7$ and different values of δ , whereas B_0 in each case is given.

4. An approximate transformation of a lowpass poles distribution of order N to the corresponding bandpass one of order $2N$ is explained in the following example:

Required is the approximate computation of the bandpass poles of a transfer function where

a- The denominator polynomial order $N=6$,

- b- The phase is increasing linearly with a ripple of an equal magnitude $\delta=0.05$, and
- c- The approximated bandpass phase function possesses a delay-center frequency product $T_m \omega_o = 9.846$.

The order of the lowpass equiripple phase function $=N/2 = 3$.

For equiripple phase approximation for $\delta=0.05$, the poles are:

(appendix V, page V.5)

δ	RE	$\pm j\text{IM}$	$T_m \omega_g$
0.05	-3.07618573799'-001 -3.47777616662'-001	7.23966325735'-001	3.413

To achieve a delay-center frequency product $T_m \omega_o$, the lowpass poles are to be scaled by $(T_m \omega_g)/(T_m \omega_o)$. Then the approximate bandpass poles are obtained by adding $\pm j$ giving

RE	$\pm j\text{IM}$
-1.06632357532'-001	1.25095446541'+000
-1.06632357532'-001	7.49045595143'-001
-1.20553017045'-001	1.00000000000'+000

The exact poles of the equiripple phase function for $N=6$, $\delta=0.05$ and $T_m \omega_o = 9.846$ are: (Appendix VI, page VI.8)

γ/k	RE	$\pm j\text{IM}$	$T_m \omega_o$
0.7	-1.07476701309'-001	1.25329623558'+000	9.846
1.69	-1.07993736747'-001 -1.21952592192'-001	7.47390388104'-001 1.00065243756'+000	

By comparing those exact and approximate solutions it is noticed that the two most significant decimal digits are identical.

This is found to be true in all the carried out examples.

This method is similar to that proposed in Eq. (4.11). Since the delay-center frequency product is the decisive factor in the delay bandpass transformation, the scaling factor is given by $(T_m \omega_g / T_m \omega_o)$ instead of $\gamma/2$ in Eq. (4.11)

If the scaling factors in both procedures are compared, then for the

above example

$$\gamma = 2 \cdot (T_m \omega_g) / (T_m \omega_o) = 0.6932 \quad , \quad (4.23)$$

which is very near to the exact value ($\gamma \approx 0.7$, see the above exact solution).

4.4.3 Results and compilation of tables

In view of the preceding properties, the frequency normalized poles of $H(p)$ are computed for $N=2,4,6,8$ and five values of δ (0.01, 0.03, 0.05, 0.07, 0.1). For each ripple magnitude, B_o is calculated to get $\gamma = 0.9, 0.7, 0.5, 0.3^{\dagger\dagger}$ respectively. This is carried out by an iterative scheme and to save time, B_o is changed in steps (0.01 $\cdot \pi$). The solution nearest to the required values of γ are tabulated. The exact values of γ , B_o and $T_m \omega_o$ in each case are also given in the tables.

4.5 Example: Calculation of Ladder Network which possesses an Equi-ripple Phase Characteristic

Data: Delay Time 125 μsec

Approximation interval $\approx 0 \dots 12$ KHZ

phase ripple ≤ 0.05

load and source resis-

tance $R_2 = R_1 = 600 \text{ ohm}$

Solution:

The delay-bandwidth product is given by

$$\begin{aligned} T_m \omega_g &= 2\pi \cdot 12 \cdot 10^3 \cdot 125 \cdot 10^{-6} \\ &= 9.425 \end{aligned} \quad (4.24)$$

For the phase approximation, Appendix V (Type III) will be considered. From these tables, to achieve the required delay-bandwidth

†† For $N=8$, $\gamma=0.9, 0.7, 0.5$ or 0.6 are only given.

product and the ripple magnitude, we found that $N = 7$ can meet all these requirements.

From the table:

	RE	$\pm IM$	$T_m \omega_g$
0.05	-1.19334573166'-001	8.94266250605'-001	9.244
	-1.40215939304'-001	6.10824759162'-001	
	-1.45636727175'-001	3.08334588597'-001	
	-1.46036506677'-001		

To achieve the exact value of the mean delay time (125 μsec), the approximation interval will be

$$f_g = \frac{9.244}{2\pi \cdot 125 \cdot 10^{-6}} = 11.769 \text{ KHZ} \quad (4.25)$$

which is as close as possible to the required bandwidth. The low-pass transfer function $H(p)$ can be written as

$$H(p) = \frac{k}{\prod_{i=1}^7 (p - p_i)} \quad (4.26)$$

This transfer function can be realized by a double resistively terminated LC ladder network, Fig. (4-5).

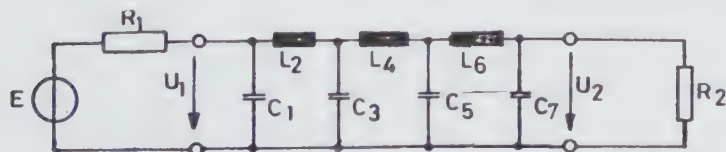


Fig.(4-5) The LC ladder network realizing the given transfer function.

In this case

$$H(p) = U_2/E, \quad (4.27)$$

and for $R_1 = R_2$

$$H(0) = 1/2 = \frac{k}{\prod_{i=1}^7 (-p_i)} \quad (4.28)$$

which gives

$$K = \frac{1}{2} \prod_{i=1}^7 (-p_i) \quad . \quad (4.29)$$

The input impedance rational function is formed and through its continued fraction expansion the normalized filter elements are calculated. The elements are denormalized with respect to the required impedance (600 Ω) and the cutoff frequency f'_g and given by:

$$C_1 = 197.3429854 \quad \text{nF}$$

$$L_2 = 16.38931950 \quad \text{mH}$$

$$C_3 = 73.75577291 \quad \text{nF}$$

$$L_4 = 12.76866150 \quad \text{mH}$$

$$C_5 = 54.89685416 \quad \text{nF}$$

$$L_6 = 11.28839751 \quad \text{mH}$$

$$C_7 = 26.73174375 \quad \text{nF}$$

4.6 Practical Implementation and Comparison between Calculated and Measured Results

For the previous calculated filter, the phase between the input and output voltage is measured and plotted (dotted line) in Fig. (4.6). For both ideal (lossless) and lossy elements the phase is calculated and plotted in the same figure, continuance line and dash-dotted line respectively. The element values are rounded to 4 decimal digits throughout the calculation.

* * *

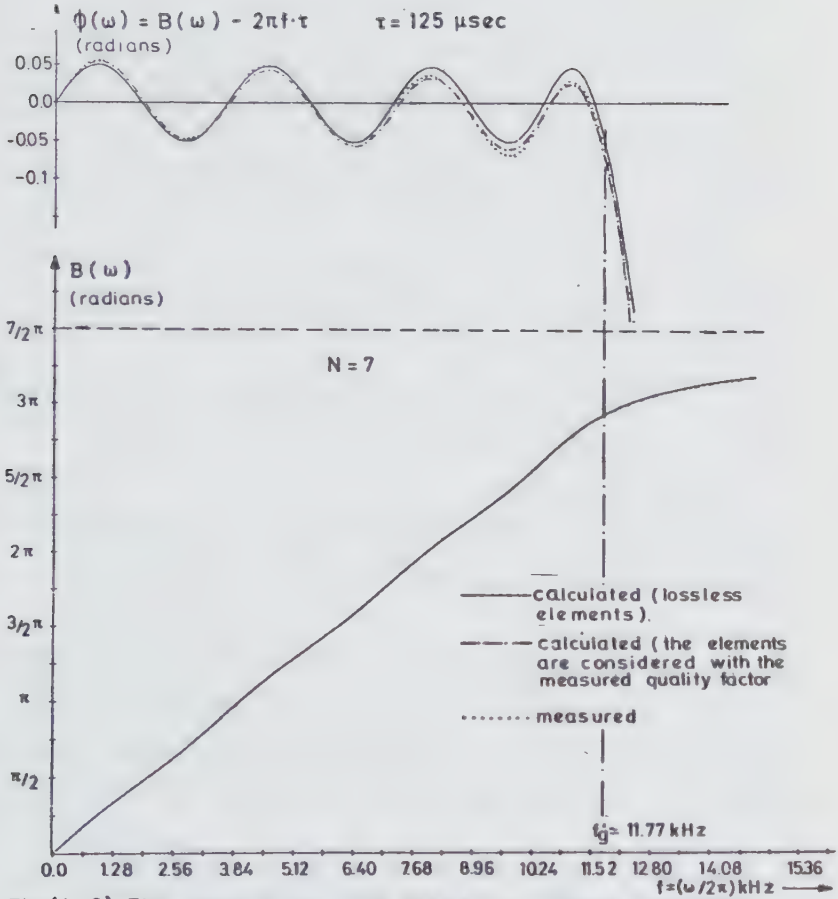


Fig (4-6) The calculated and measured Phase functions $B(\omega)$ and $\phi(\omega)$.

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Appendices I-VI

APPENDIX I

Proof of the expression $T(\omega) = \sum_{i=1}^N \frac{-p_i}{(\omega^2 + p_i^2)}$

Let the all-pass network function be

$$H_a(p) = \frac{Q(-p)}{Q(p)} = \prod_{i=1}^N \frac{-(p+p_i)}{(p-p_i)} = e^{-\Gamma} \quad (I-1)$$

For the lowpass function

$$H(p) = \frac{1}{Q(p)} \bigg|_{p=j\omega} = \left| H(j\omega) \right| e^{-jB(\omega)} \quad (I-2)$$

where $B(\omega)$ is the phase function.

From Eq. (I-1) and Eq. (I-2)

$$2j B(\omega) = \Gamma \bigg|_{p=j\omega} \quad (I-3)$$

or the delay time

$$T(\omega) = \frac{\partial B(\omega)}{\partial \omega} = \frac{1}{2} \frac{\partial \Gamma}{\partial p} \bigg|_{p=j\omega} \quad (I-4)$$

but from Eq. (I-1)

$$\Gamma = -\ln(-1)^N + \sum_{i=1}^N \ln(p-p_i) - \sum_{i=1}^N \ln(p+p_i) \quad (I-5)$$

and

$$\frac{\partial \Gamma}{\partial p} = \sum_{i=1}^N \left(\frac{1}{(p-p_i)} - \frac{1}{(p+p_i)} \right) \quad (I-6)$$

Substitute Eq. (I-6) in Eq. (I-4), then

$$T(\omega) = \sum_{i=1}^N \frac{-p_i}{(\omega^2 + p_i^2)} \quad (I-7)$$

APPENDIX II

Slope of $V(\omega)$ at the origin (SL)

Consider $H(p)$ is given as:

$$H(p) = \frac{K}{1+a_1p+a_2p^2+\dots+a_Np^N} \quad (II-1)$$

or for $p=j\omega$

$$H(p) = \frac{K}{M(\omega) + jN(\omega)} \quad (II-2)$$

where

$$M(\omega) = 1 - a_2\omega^2 + a_4\omega^4 - a_6\omega^6 \dots$$

$$N(\omega) = a_1\omega - a_3\omega^3 + a_5\omega^5 - \dots$$

The values of $M(\omega)$ and its derivatives at $\omega = 0$ are:

$$\begin{aligned} M(0) &= 1 & M'(0) &= 0 \\ M''(0) &= -2a_2 & M'''(0) &= 0 \end{aligned} \quad (II-3)$$

And also for $N(\omega)$

$$\begin{aligned} N(0) &= 0 & N'(0) &= a_1 \\ N''(0) &= 0 & N'''(0) &= -6a_3 \end{aligned} \quad (II-4)$$

The delay time function of $H(p)$ Eq. (II-2) is

$$T(\omega) = \frac{MN' - NM'}{M^2 + N^2} \quad (II-5)$$

then $V(\omega) = \frac{\partial T(\omega)}{\partial \omega}$ will be

$$V(\omega) = \frac{MN'' - NM''}{M^2 + N^2} - T(\omega) \frac{2(MM' + NN')}{M^2 + N^2} \quad (II-6)$$

For convenience we write

$$V(\omega) = \frac{C(\omega)}{Q(\omega)} \quad (II-7)$$

The derivative of $V(\omega)$ w. r. t. ω is

$$V'(\omega) = \frac{C'(\omega)}{Q(\omega)} - \frac{V(\omega)Q'(\omega)}{Q(\omega)} \quad (II-8)$$

At origin i. e. $\omega = 0$

$$T(0) = a_1, \quad V(0) = 0, \quad Q(0) = 1 \quad (II-9)$$

Then

$$V'(0) = C'(0) \quad (II-10)$$

But from Eq. (II-6)

$$\begin{aligned} C'(\omega) = & MN''' + M'N'' - NM''' - N'M'' \\ & - 2 T(\omega)(MM'' + M'^2 + N'^2 + NN'') \\ & - 2 V(\omega)(MM' + NN') \end{aligned} \quad (II-11)$$

Substitute the values of M , N and its derivatives at the origin

' Eqs.(II-3,4) and using Eqs.(II-9,10), the slope at the origin is

$$SL = 6(a_1 a_2 - a_3) - 2(a_1)^3 \quad (II-12)$$

Eq. (II-12) is the slope of $V(\omega)$ at $\omega = 0$ for any value of polynomial order N .

APPENDIX III

Solution of the Equiripple Approximation of the delay time $T(\omega)$

"TYPE I"

1. Relations

$$T(\omega) = \sum_{i=1}^N \frac{-p_i}{(\omega^2 + p_i^2)}$$

In the approximation interval $T(\omega)$ has N peaks at which

$$T(\omega) = T_m (1 \pm \delta)$$

2. Parameters

Mean delay time

T_m

Upper edge frequency

ω_g

Delay-bandwidth product

$T_m \omega_g$

Normalized frequency

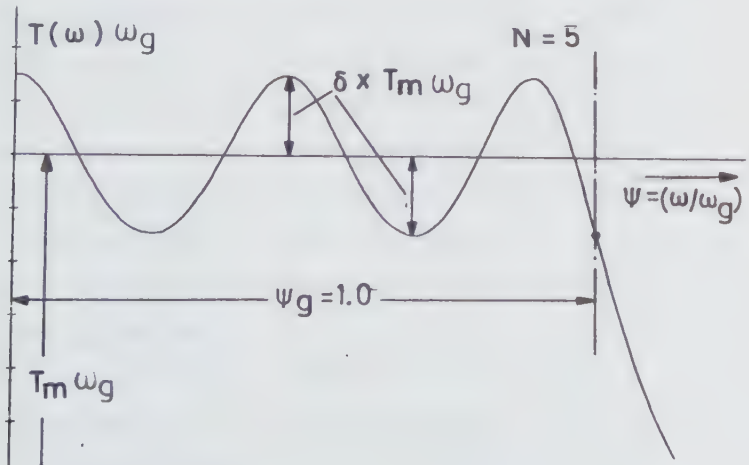
$\Psi = \omega / \omega_g$

Normalized poles

$RE \pm j IM = (p_i / \omega_g)$

Normalized ripple magnitude

$\delta \cdot (T_m \omega_g)$



Frequency normalized $T(\omega)$

Type I	N=2	$\Psi_g = 1.0$	
δ	RE	$\pm IM$	$T_m \omega_g$
0.010	-1.49656304653'+000	1.03918252356'+000	0.911
0.020	-1.21369263838'+000	9.07937189546'-001	1.078
0.030	-1.06602808792'+000	8.43883065478'-001	1.189
0.040	-9.68202801159'-001	8.03620280232'-001	1.274
0.050	-8.95937075889'-001	7.75178096475'-001	1.344
0.060	-8.39068331847'-001	7.53665634435'-001	1.403
0.070	-7.92421487931'-001	7.36643698288'-001	1.456
0.080	-7.53020115073'-001	7.22735381201'-001	1.503
0.090	-7.19002878668'-001	7.11094259928'-001	1.545
0.100	-6.89132132276'-001	7.01166431900'-001	1.584
0.110	-6.62545333282'-001	6.92571613326'-001	1.621
0.120	-6.38618569337'-001	6.85038606465'-001	1.655
0.130	-6.16986696429'-001	6.78367942481'-001	1.687
0.140	-5.96994144700'-001	6.72409103923'-001	1.717
0.150	-5.78663312561'-001	6.67046032701'-001	1.746
0.160	-5.61673528915'-001	6.62187574915'-001	1.774
0.170	-5.45846624311'-001	6.57760984293'-001	1.800
0.180	-5.31036774799'-001	6.53707390244'-001	1.826
0.190	-5.17123190377'-001	6.49978562207'-001	1.851
0.200	-5.03992744772'-001	6.46531434340'-001	1.875
0.210	-4.91595974621'-001	6.43341952153'-001	1.898
0.220	-4.79824013186'-001	6.40372569571'-001	1.921
0.230	-4.68626268095'-001	6.37602430754'-001	1.944
0.240	-4.57948583731'-001	6.35010998674'-001	1.966

Type I	N= 2	$\Psi_g = 1.0$	
δ	RE	$\pm IM$	$T_m \omega_g$
0.250	-4.47743802371'-001	6.32580564721'-001	1.988
0.260	-4.37970616149'-001	6.30295766710'-001	2.009
0.270	-4.28592643333'-001	6.28143203914'-001	2.031
0.280	-4.19577678447'-001	6.26111126487'-001	2.052
0.290	-4.10897078600'-001	6.24189183149'-001	2.073
0.300	-4.02525257541'-001	6.22368214386'-001	2.093

Type I	N=3	$\Psi_g = 1.0$	
δ	RE	$\pm 1M$	$T_m \omega_g$
0.010	-7.45532640860'-001 -8.96315245715'-001	1.03233789117'+000	2.015
0.020	-6.29062757442'-001 -7.44794073689'-001	9.57930029326'-001	2.255
0.030	-5.65011173141'-001 -6.61597358558'-001	9.20045605617'-001	2.409
0.040	-5.21168622290'-001 -6.04838735308'-001	8.95538572244'-001	2.523
0.050	-4.87991292994'-001 -5.62060881726'-001	8.77831600596'-001	2.616
0.060	-4.61377081931'-001 -5.27896358253'-001	8.64183266414'-001	2.694
0.070	-4.39194922594'-001 -4.99551804474'-001	8.53204927480'-001	2.762
0.080	-4.20199233158'-001 -4.75392356588'-001	8.44102456747'-001	2.823
0.090	-4.03600467154'-001 -4.54380660900'-001	8.36382138579'-001	2.878
0.100	-3.88867469475'-001 -4.35818322493'-001	8.29717612316'-001	2.928
0.110	-3.75626240213'-001 -4.19213310699'-001	8.23882752855'-001	2.975
0.120	-3.63603751521'-001 -4.04206295578'-001	8.18714933761'-001	3.018
0.130	-3.52594752319'-001 -3.90527085197'-001	8.14093597852'-001	3.059
0.140	-3.42441145003'-001 -3.77967539491'-001	8.09927100079'-001	3.098
0.150	-3.33018630427'-001 -3.66364018075'-001	8.06144282047'-001	3.135
0.160	-3.24227755424'-001 -3.55585606863'-001	8.02688882624'-001	3.170
0.170	-3.15987732054'-001 -3.45525984951'-001	7.99515717030'-001	3.204

Type I	N=3	$\psi_g = 1.0$	
δ	RE	$\pm IM$	$T_m \omega_g$
0.180	-3.08232059866'-001 -3.36097662148'-001	7.96587996319'-001	3.238
0.190	-3.00905357042'-001 -3.27227805217'-001	7.93875404041'-001	3.270
0.200	-2.93961020947'-001 -3.18855155667'-001	7.91352688637'-001	3.301
0.210	-2.87359470927'-001 -3.10927714236'-001	7.88998615109'-001	3.332
0.220	-2.81066808620'-001 -3.03400974597'-001	7.86795171840'-001	3.362
0.230	-2.75053781730'-001 -2.96236557209'-001	7.84726961862'-001	3.391
0.240	-2.69294972946'-001 -2.89401139598'-001	7.82780729936'-001	3.420
0.250	-2.63768157700'-001 -2.82865609246'-001	7.80944991002'-001	3.449
0.260	-2.58453790306'-001 -2.76604385814'-001	7.79209734388'-001	3.478
0.270	-2.53334588942'-001 -2.70594873640'-001	7.77566187328'-001	3.506
0.280	-2.48395197694'-001 -2.64817015937'-001	7.76006622935'-001	3.535
0.290	-2.43621908907'-001 -2.59252928336'-001	7.74524204117'-001	3.563
0.300	-2.39002433515'-001 -2.53886596154'-001	7.73112854742'-001	3.591

Type I	N = 4	$\Psi_g = 1.0$	
δ	RE	$\pm 1M$	$T_m \omega_g$
0.010	-4.79545030088'-001 -6.01152475792'-001	1.02181381779'+000 3.50717432386'-001	3.268
0.020	-4.12642281146'-001 -5.06069460418'-001	9.72366961953'-001 3.35063859187'-001	3.558
0.030	-3.74846567622'-001 -4.52813025953'-001	9.46528565051'-001 3.26695237978'-001	3.740
0.040	-3.48523907927'-001 -4.16048336003'-001	9.29516376240'-001 3.21064309785'-001	3.875
0.050	-3.28343706151'-001 -3.88105322276'-001	9.17054851620'-001 3.16855554534'-001	3.983
0.060	-3.11984591830'-001 -3.65642281708'-001	9.07340067442'-001 3.13512583580'-001	4.075
0.070	-2.98228546395'-001 -3.46905902865'-001	8.99449385757'-001 3.10749815559'-001	4.154
0.080	-2.86357871268'-001 -3.30863437863'-001	8.92850860077'-001 3.08401899500'-001	4.224
0.090	-2.75914506812'-001 -3.16856125701'-001	8.87211415793'-001 3.06364812986'-001	4.287
0.100	-2.66588458751'-001 -3.04438472056'-001	8.82309478639'-001 3.04569006017'-001	4.345
0.110	-2.58160295431'-001 -2.93295401381'-001	8.77990639026'-001 3.02965753617'-001	4.399
0.120	-2.50469062423'-001 -2.83196027263'-001	8.74143257439'-001 3.01519633791'-001	4.450
0.130	-2.43393234927'-001 -2.73966208708'-001	8.70684165700'-001 3.00204109797'-001	4.497
0.140	-2.36838831861'-001 -2.65471427229'-001	8.67549849518'-001 2.98998799983'-001	4.543
0.150	-2.30731689615'-001 -2.57605658015'-001	8.64690778853'-001 2.97887718247'-001	4.586
0.160	-2.25012263178'-001 -2.50283883826'-001	8.62067631988'-001 2.96858099417'-001	4.627

Type I	N= 4	$\Psi_g = 1.0$	
δ	RE	$\pm IM$	$T_m \omega_g$
0.170	-2.19632025146'-001 -2.43436910710'-001	8.59648705566'-001 2.95899591190'-001	4.668
0.180	-2.14550907334'-001 -2.37007686279'-001	8.57408090964'-001 2.95003683888'-001	4.707
0.190	-2.09735443244'-001 -2.30948627097'-001	8.55324361870'-001 2.94163298797'-001	4.745
0.200	-2.05157392942'-001 -2.25219641423'-001	8.53379610704'-001 2.93372485245'-001	4.782
0.210	-2.00792707645'-001 -2.19786640948'-001	8.51558728835'-001 2.92626193390'-001	4.818
0.220	-1.96620738243'-001 -2.14620403189'-001	8.49848860522'-001 2.91920101539'-001	4.854
0.230	-1.92623621737'-001 -2.09695690639'-001	8.48238982686'-001 2.91250482809'-001	4.890
0.240	-1.88785799874'-001 -2.04990559607'-001	8.46719577556'-001 2.90614100923'-001	4.925
0.250	-1.85093637083'-001 -2.00485812376'-001	8.45282374699'-001 2.90008127831'-001	4.960
0.260	-1.81535114414'-001 -1.96164558023'-001	8.43920145639'-001 2.89430077894'-001	4.995
0.270	-1.78099581575'-001 -1.92011857245'-001	8.42626538466'-001 2.88877754902'-001	5.030
0.280	-1.74777554708'-001 -1.88014432864'-001	8.41395944095'-001 2.88349208709'-001	5.065
0.290	-1.71560550112'-001 -1.84160431541'-001	8.40223386498'-001 2.87842699985'-001	5.100
0.300	-1.68440946200'-001 -1.80439226588'-001	8.39104432656'-001 2.87356670884'-001	5.135

Type 1	N= 5	$\Psi_g = 1.0$	
δ	RE	$\pm 1M$	$T_m \omega_g$
0.010	-3.48368219231'-001 -4.43084534491'-001 -4.62407774961'-001	1.01585703390'+000 5.24655847650'-001	4.600
0.020	-3.03475849029'-001 -3.76535008254'-001 -3.90293382836'-001	9.79357484463'-001 5.07248347072'-001	4.928
0.030	-2.77616015068'-001 -3.38760907573'-001 -3.49734027128'-001	9.59998094491'-001 4.97533347229'-001	5.131
0.040	-2.59401437499'-001 -3.12468191205'-001 -3.21648116122'-001	9.47101818139'-001 4.90877962486'-001	5.281
0.050	-2.45317439214'-001 -2.92361523396'-001 -3.00259997297'-001	9.37570002301'-001 4.85837694927'-001	5.401
0.060	-2.33820839098'-001 -2.76118403494'-001 -2.83042820884'-001	9.30084343734'-001 4.81793108345'-001	5.502
0.070	-2.24096829294'-001 -2.62514128655'-001 -2.68667070908'-001	9.23966135862'-001 4.78422719993'-001	5.590
0.080	-2.15662821909'-001 -2.50824374256'-001 -2.56348010764'-001	9.18821909116'-001 4.75538647166'-001	5.667
0.090	-2.08209462664'-001 -2.40585524858'-001 -2.45584285905'-001	9.14404113402'-001 4.73021733214'-001	5.738
0.100	-2.01526592882'-001 -2.31483121246'-001 -2.36036423921'-001	9.10547366577'-001 4.70791737804'-001	5.802
0.110	-1.95464943190'-001 -2.23294175437'-001 -2.27464087511'-001	9.07135973914'-001 4.68792086002'-001	5.862
0.120	-1.89914686798'-001 -2.15854895625'-001 -2.19691009028'-001	9.04085992152'-001 4.66981410336'-001	5.918
0.130	-1.84792668144'-001 -2.09041479211'-001 -2.12584147375'-001	9.01334681988'-001 4.65328555983'-001	5.971

Type I	N=5	$\Psi_g = 1.0$	
δ	RE	$\pm IM$	$T_m \omega_g$
0.140	-1.80034412322'-001 -2.02758103475'-001 -2.06040669895'-001	8.98833984975'-001 4.63809478395'-001	6.022
0.150	-1.75588917049'-001 -1.96929104732'-001 -1.99979483266'-001	8.96546317112'-001 4.62405233953'-001	6.070
0.160	-1.71415140903'-001 -1.91493707400'-001 -1.94335531989'-001	8.94441760836'-001 4.61100633038'-001	6.117
0.170	-1.67479564934'-001 -1.86402368399'-001 -1.89055847019'-001	8.92496133223'-001 4.59883309562'-001	6.162
0.180	-1.63754457695'-001 -1.81614176297'-001 -1.84096738274'-001	8.90689624132'-001 4.58743061987'-001	6.206
0.190	-1.60216612119'-001 -1.77094961109'-001 -1.79421755158'-001	8.89005812715'-001 4.57671375305'-001	6.249
0.200	-1.56846409653'-001 -1.72815893460'-001 -1.75000176994'-001	8.87430944666'-001 4.56661068991'-001	6.292
0.210	-1.53627113999'-001 -1.68752429137'-001 -1.70805876274'-001	8.85953390909'-001 4.55706031091'-001	6.334
0.220	-1.50544330561'-001 -1.64883501362'-001 -1.66816449710'-001	8.84563236483'-001 4.54801016334'-001	6.375
0.230	-1.47585586554'-001 -1.61190894371'-001 -1.63012544617'-001	8.83251962998'-001 4.53941489250'-001	6.416
0.240	-1.44740001138'-001 -1.57658751665'-001 -1.59377330681'-001	8.82012201688'-001 4.53123501422'-001	6.457
0.250	-1.41998023624'-001 -1.54273185428'-001 -1.55896080702'-001	8.80837537654'-001 4.52343594268'-001	6.498

Type I	N=5	$\psi_g = 1.0$	
δ	RE	$\pm 1M$	$T_m \omega_g$
0.260	-1.39351222857'-001 -1.51021963477'-001 -1.52555834786'-001	8.79722354373'-001 4.51598721127'-001	6.538
0.270	-1.36792116835'-001 -1.47894255639'-001 -1.49345128470'-001	8.78661707990'-001 4.50886184062'-001	6.579
0.280	-1.34314033461'-001 -1.44880426768'-001 -1.46253771040'-001	8.77651225594'-001 4.50203582539'-001	6.620
0.290	-1.31910995755'-001 -1.41971866498'-001 -1.43272663241'-001	8.76687021555'-001 4.49548771339'-001	6.661
0.300	-1.29577626536'-001 -1.39160847884'-001 -1.40393646158'-001	8.75765628782'-001 4.48919825401'-001	6.703

Type I	N=6	$\Psi_g = 1.0$	
δ	RE	$\pm IM$	$T_m \omega_g$
0.010	-2.71755951861'-001 -3.47709243120'-001 -3.68144708220'-001	1.01211949240'+000 6.27838013796'-001 2.12424788260'-001	5.980
0.020	-2.97545605736'-001 -2.38620825126'-001 -3.12198496191'-001	6.10795804485'-001 9.83528505301'-001 2.06495928745'-001	6.338
0.030	-2.68799009013'-001 -2.19337103597'-001 -2.80482196034'-001	6.01258213199'-001 3.68135238556'-001 2.03122796564'-001	6.559
0.040	-2.05643744092'-001 -2.48663166753'-001 -2.58437858960'-001	9.57796093539'-001 5.94651566225'-001 2.00763054876'-001	6.721
0.050	-1.94990328157'-001 -2.33191798841'-001 -2.41604005976'-001	9.50106101911'-001 5.89608417406'-001 1.98949652397'-001	6.850
0.060	-1.86250544575'-001 -2.20645598925'-001 -2.28022728083'-001	9.44035929089'-001 5.85536848041'-001 1.97478510629'-001	6.959
0.070	-1.78827041738'-001 -2.10103954090'-001 -2.16661508734'-001	9.39053135917'-001 5.82127404930'-001 1.96242128797'-001	7.053
0.080	-1.72364730588'-001 -2.01020717854'-001 -2.06909800254'-001	9.34847831843'-001 5.79198144093'-001 1.95176901114'-001	7.137
0.090	-1.66635277856'-001 -1.93045410324'-001 -1.98376939508'-001	9.31224385236'-001 5.76633082033'-001 1.94242061396'-001	7.213
0.100	-1.61483143365'-001 -1.85939768402'-001 -1.90798056054'-001	9.28051704490'-001 5.74353766457'-001 1.93409905271'-001	7.282
0.110	-1.56797583430'-001 -1.79534495561'-001 -1.83985412245'-001	9.25237833362'-001 5.72304664936'-001 1.92660741217'-001	7.347
0.120	-1.52496937801'-001 -1.73705010708'-001 -1.77801174788'-001	9.22715886412'-001 5.70445028261'-001 1.91980067026'-001	7.408

Type 1	N=6	$\psi_g = 1.0$	
δ	RE	$\pm IM$	$T_m \omega_g$
0.130	-1.48519252914'-001 -1.68356992799'-001 -1.72141213858'-001	9.20405758810'-001 5.68744069809'-001 1.91356890166'-001	7.466
0.140	-1.63417325730'-001 -1.44816404092'-001 -1.66924974183'-001	5.67177959780'-001 9.18358988311'-001 1.90782677350'-001	7.521
0.150	-1.41350256705'-001 -1.58828193438'-001 -1.62088879388'-001	9.16455435894'-001 5.65727873764'-001 1.90250669868'-001	7.574
0.160	-1.38090074499'-001 -1.54543094657'-001 -1.57581888736'-001	9.14701065118'-001 5.64378680771'-001 1.89755422212'-001	7.626
0.170	-1.50524075147'-001 -1.35010716535'-001 -1.53362415327'-001	5.63118034570'-001 9.13076412697'-001 1.89292481417'-001	7.676
0.180	-1.46739756357'-001 -1.32091352407'-001 -1.49396134464'-001	5.61935728118'-001 9.11565507588'-001 1.88858158883'-001	7.725
0.190	-1.43163901234'-001 -1.29314526654'-001 -1.45654391392'-001	5.60823225635'-001 9.10155090249'-001 1.88449364392'-001	7.773
0.200	-1.39774349707'-001 -1.26665463136'-001 -1.42113021368'-001	5.59773314901'-001 9.08834037331'-001 1.88063482772'-001	7.820
0.210	-1.36552216573'-001 -1.24131541081'-001 -1.38751460960'-001	5.58779847193'-001 9.07592930482'-001 1.87698280941'-001	7.867
0.220	-1.33481276465'-001 -1.21701891789'-001 -1.35552068110'-001	5.57837536428'-001 9.06423728164'-001 1.87351836645'-001	7.913
0.230	-1.19367086860'-001 -1.30547486711'-001 -1.32499595133'-001	9.05319512334'-001 5.56941805000'-001 1.87022483151'-001	7.959
0.240	-1.17118891933'-001 -1.29580774843'-001 -1.27738612340'-001	9.04274290755'-001 1.86708765636'-001 5.56088662335'-001	8.005

Type I	N= 6	$\psi_g = 1.0$	
δ	RE	$\pm IM$	$T_m \omega_g$
0.250	-1.14950070951'-001 -1.26783992189'-001 -1.25043928154'-001	9.03282840227'-001 1.86409406630'-001 5.55274607516'-001	8.051
0.260	-1.12854229410'-001 -1.24099021079'-001 -1.22453979349'-001	9.02340582343'-001 1.86123278232'-001 5.54496552762'-001	8.098
0.270	-1.10825686497'-001 -1.21516811192'-001 -1.19960388111'-001	9.01443481995'-001 1.85849379515'-001 5.53751760461'-001	8.144
0.280	-1.08859371059'-001 -1.19029314010'-001 -1.17555695004'-001	9.00587966138'-001 1.85586818090'-001 5.53037791731'-001	8.191
0.290	-1.06950735608'-001 -1.16629339784'-001 -1.15233228684'-001	8.99770856333'-001 1.85334794907'-001 5.52352464641'-001	8.238
0.300	-1.05095684801'-001 -1.14310438945'-001 -1.12986997723'-001	8.98989313580'-001 1.85092591706'-001 5.51693819027'-001	8.286

Type I	N= 7	$\Psi_g = 1.0$	
δ	RE	$\pm IM$	$T_m \omega_g$
0.010	-2.84765429868'-001 -2.21860948841'-001 -3.03493126095'-001 -3.08047211176'-001	6.95249954722'-001 1.00975132955'+000 3.53354447252'-001	7.394
0.020	-2.44990024981'-001 -1.95979596476'-001 -2.58441989405'-001 -2.61638392462'-001	6.79406252424'-001 9.86308349013'-001 3.44903534246'-001	7.776
0.030	-1.80774899560'-001 -2.22019933892'-001 -2.32759993218'-001 -2.35279323359'-001	9.73569144677'-001 6.70430440237'-001 3.40033341992'-001	8.011
0.040	-1.69911148743'-001 -2.14845643826'-001 -2.05850024144'-001 -2.16937555015'-001	9.64960175115'-001 3.36601246429'-001 6.64166485534'-001	8.182
0.050	-1.93379431021'-001 -1.61419496606'-001 -2.01128576802'-001 -2.02919156498'-001	6.59359630263'-001 9.58527278453'-001 3.33950641347'-001	8.320
0.060	-1.83236220575'-001 -1.54426654586'-001 -1.90037623397'-001 -1.91601436723'-001	6.55463155657'-001 9.53430284371'-001 3.31792413301'-001	8.435
0.070	-1.74692091430'-001 -1.48467902585'-001 -1.80742473582'-001 -1.82128064320'-001	6.52189792480'-001 9.49233064555'-001 3.29973400963'-001	8.535
0.080	-1.67313942147'-001 -1.43266228752'-001 -1.72751304883'-001 -1.73992405209'-001	6.49369944324'-001 9.45681035664'-001 3.28402574785'-001	8.624
0.090	-1.60823266608'-001 -1.38643085560'-001 -1.65748962420'-001 -1.66870147652'-001	6.46895122645'-001 9.42613072417'-001 3.27021377016'-001	8.705
0.100	-1.55030392045'-001 -1.34476595769'-001 -1.59521498630'-001 -1.60541328798'-001	6.44691724679'-001 9.39920963134'-001 3.25789888389'-001	8.779

Type I	N=7	$\Psi_g = 1.0$	
δ	RE	$\pm IM$	$T_m \omega_g$
0.110	-1.49800331619'-001 -1.30679823833'-001 -1.53917097741'-001 -1.54849992530'-001	6.42707512263'-001 9.37528647741'-001 3.24679659469'-001	8.848
0.120	-1.45033609504'-001 -1.27188558421'-001 -1.48824185817'-001 -1.49681607774'-001	6.40904064319'-001 9.35380700877'-001 3.23669689016'-001	8.914
0.130	-1.40654812534'-001 -1.23954010152'-001 -1.44158392788'-001 -1.44949627575'-001	6.39252290850'-001 9.33435532632'-001 3.22744025729'-001	8.976
0.140	-1.36605411270'-001 -1.20938226453'-001 -1.39854393452'-001 -1.40587083704'-001	6.37729632232'-001 9.31661166622'-001 3.21890263818'-001	9.035
0.150	-1.32839071786'-001 -1.18111094253'-001 -1.35860590308'-001 -1.36541112152'-001	6.36318236226'-001 9.30032509590'-001 3.21098561290'-001	9.093
0.160	-1.29318489002'-001 -1.15448312401'-001 -1.32135528511'-001 -1.32769263968'-001	6.35003730029'-001 9.28529520930'-001 3.20360976446'-001	9.148
0.170	-1.26013184243'-001 -1.12929981086'-001 -1.28645407942'-001 -1.29236946315'-001	6.33774368231'-001 9.27135952596'-001 3.19671006215'-001	9.203
0.180	-1.10539594076'-001 -1.25362312374'-001 -1.22897936002'-001 -1.25915602333'-001	9.25838455045'-001 3.19023256976'-001 6.32620427259'-001	-9.256
0.190	-1.08263304723'-001 -1.22262922112'-001 -1.19951637051'-001 -1.22781388236'-001	9.24625931879'-001 3.18413204826'-001 6.31533764892'-001	9.308
0.200	-1.06089380650'-001 -1.19327559605'-001 -1.17156444918'-001 -1.19814193218'-001	9.23489064349'-001 3.17837017297'-001 6.30507493373'-001	9.360

Type I	N= 7	$\Psi_g = 1.0$	
δ	RE	$\pm IM$	$T_m \omega_g$
0.210	-1.04007790778'-001 -1.16539470946'-001 -1.14497141454'-001 -1.16996900961'-001	9.22419953509'-001 3.17291418432'-001 6.29535733431'-001	9.412
0.220	-1.02009888815'-001 -1.13884273345'-001 -1.11960642109'-001 -1.14314824576'-001	9.21411847783'-001 3.16773585746'-001 6.28613425633'-001	9.463
0.230	-1.09535614123'-001 -1.00088167503'-001 -1.11349530118'-001 -1.11755268597'-001	6.27736184350'-001 9.20458933975'-001 3.16281069834'-001	9.514
0.240	-1.07212176394'-001 -9.82360645685'-002 -1.08924414286'-001 -1.09307184946'-001	6.26900182051'-001 9.19556171208'-001 3.15811730881'-001	9.565
0.250	-1.04981661058'-001 -9.64478084970'-002 -1.06599442330'-001 -1.06960900004'-001	6.26102058195'-001 9.18699162197'-001 3.15363688511'-001	9.616
0.260	-1.02836422280'-001 -9.47182945893'-002 -1.04366260781'-001 -1.04707895676'-001	6.25338846428'-001 9.17884048111'-001 3.14935281247'-001	9.668
0.270	-1.00769680317'-001 -9.30429845321'-002 -1.02217473119'-001 -1.02540632259'-001	6.24607914207'-001 9.17107426434'-001 3.14525033813'-001	9.720
0.280	-9.87753948903'-002 -9.14178234780'-002 -1.00146498484'-001 -1.00452403673'-001	6.23906914716'-001 9.16366280428'-001 3.14131629799'-001	9.772
0.290	-9.68481605378'-002 -8.98391726769'-002 -9.81474556611'-002 -9.84372185994'-002	6.23233746595'-001 9.15657925492'-001 3.13753890012'-001	9.825
0.300	-9.49831197104'-002 -8.83037525455'-002 -9.62150664673'-002 -9.64897017402'-002	6.22586520876'-001 9.14979961053'-001 3.13390753639'-001	9.879

Type I	N=7	Ψ_g 1.0	
δ	RE	$\pm$ IM	$T_m \omega_g$
0.110	-1.49800331619'-001 -1.30679823833'-001 -1.53917097741'-001 -1.50849992530'-001	6.42707512263'-001 9.37528647741'-001 3.24679659469'-001	8.848
0.120	-1.45033609504'-001 -1.27188558421'-001 -1.48824185817'-001 -1.49681607774'-001	6.40904064319'-001 9.35380700877'-001 3.23669689016'-001	8.914
0.130	-1.40654812534'-001 -1.23954010152'-001 -1.44158392788'-001 -1.44949627575'-001	6.39252290850'-001 9.33435532632'-001 3.22744025729'-001	8.976
0.140	-1.36605411270'-001 -1.20938226453'-001 -1.39854393452'-001 -1.40587083704'-001	6.37729632232'-001 9.31661166622'-001 3.21890263818'-001	9.035
0.150	-1.32839071786'-001 -1.18111094253'-001 -1.35860590308'-001 -1.36541112152'-001	6.36318236226'-001 9.30032509590'-001 3.21098561290'-001	9.093
0.160	-1.29318489002'-001 -1.15448312401'-001 -1.32135528511'-001 -1.32769263968'-001	6.35003730029'-001 9.28529520930'-001 3.20360976446'-001	9.148
0.170	-1.26013184243'-001 -1.12929981086'-001 -1.28645407942'-001 -1.29236946315'-001	6.33774368231'-001 9.27135952596'-001 3.19671006215'-001	9.203
0.180	-1.10539594076'-001 -1.25362312374'-001 -1.22897936002'-001 -1.25915602333'-001	9.25838455045'-001 3.19023256976'-001 6.32620427259'-001	9.256
0.190	-1.08263304723'-001 -1.22262922112'-001 -1.19951637051'-001 -1.22781389236'-001	9.24625931879'-001 3.18413204826'-001 6.31533764892'-001	9.308
0.200	-1.06089380650'-001 -1.19327559605'-001 -1.17156444918'-001 -1.19814193218'-001	9.23489064349'-001 3.17837017297'-001 6.30507493373'-001	9.360

Type I	N = 7	$\Psi_g = 1.0$	
δ	RE	$\pm 1M$	$T_m \omega_g$
0.210	-1.04007790778'-001 -1.16539470946'-001 -1.14497141454'-001 -1.16996900961'-001	9.22419953509'-001 3.17291418432'-001 6.29535733431'-001	9.412
0.220	-1.02009888815'-001 -1.13884273345'-001 -1.11960642109'-001 -1.14314824576'-001	9.21411841783'-001 3.16773585746'-001 6.28613425633'-001	9.463
0.230	-1.09535614123'-001 -1.00088167503'-001 -1.11349530118'-001 -1.11755268597'-001	6.27736184350'-001 9.20458933975'-001 3.16281069834'-001	9.514
0.240	-1.07212176394'-001 -9.82360645685'-002 -1.08924414286'-001 -1.09307184946'-001	6.26900182051'-001 9.19556171208'-001 3.15811730881'-001	9.565
0.250	-1.04981661058'-001 -9.64478084970'-002 -1.06599442330'-001 -1.06960900074'-001	6.26102058195'-001 9.18699162197'-001 3.15363688511'-001	9.616
0.260	-1.02836422280'-001 -9.47182945893'-002 -1.04366260781'-001 -1.04707895676'-001	6.25338846428'-001 9.17884048111'-001 3.14935281247'-001	9.668
0.270	-1.00769680317'-001 -9.30429845321'-002 -1.02217473119'-001 -1.02540632259'-001	6.24607914207'-001 9.17107426434'-001 3.14525033813'-001	9.720
0.280	-9.87753948903'-002 -9.14178234780'-002 -1.00146498484'-001 -1.00452403673'-001	6.23906914716'-001 9.16366280428'-001 3.14131629799'-001	9.772
0.290	-9.68481605378'-002 -8.98391726769'-002 -9.81474556611'-002 -9.84372185994'-002	6.23233746595'-001 9.15657925492'-001 3.13753890012'-001	9.825
0.300	-9.49831197104'-002 -8.83037525455'-002 -9.62150664673'-002 -9.64897017402'-002	6.22586520876'-001 9.14979961053'-001 3.13390753639'-001	9.879

Type I	N= 8	$\Psi_g = 1.0$	
δ	RE	$\pm IM$	$T_m \omega_g$
0.010	-1.87001576433'-001 -2.57144967702'-001 -2.62757930410'-001 -2.40451147660'-001	1.00811331777'+000 4.52992069033'-001 1.52055839797'-001 7.42584117979'-001	8.831
0.020	-1.65944624201'-001 -2.23698971669'-001 -2.19758049883'-001 -2.07737840974'-001	9.88294447314'-001 1.48753204379'-001 4.43488798614'-001 7.27985437261'-001	9.234
0.030	-1.98340588620'-001 -1.53481690293'-001 -1.88728135472'-001 -2.01448666867'-001	4.37965841171'-001 9.77446999656'-001 7.19641517153'-001 1.46826564826'-001	9.480
0.040	-1.75292331551'-001 -1.44533305673'-001 -1.85935926675'-001 -1.83353478435'-001	7.13787330398'-001 9.70081821983'-001 1.45460102227'-001 4.34055074453'-001	9.660
0.050	-1.37512855679'-001 -1.64899133379'-001 -1.74062184244'-001 -1.71850348401'-001	9.64558677431'-001 7.09277917602'-001 1.44400497200'-001 4.31024959118'-001	9.804
0.060	-1.62531480251'-001 -1.31714163486'-001 -1.56425137251'-001 -1.64464336755'-001	4.28551754019'-001 9.60169884507'-001 7.05611993675'-001 1.43535294305'-001	9.925
0.070	-1.54708739185'-001 -1.26760437153'-001 -1.49272559564'-001 -1.56422260166'-001	4.26463317657'-001 9.56547078320'-001 7.02525194269'-001 1.42804568543'-001	10.030
0.080	-1.47973897481'-001 -1.22426587677'-001 -1.43085221127'-001 -1.49509526331'-001	4.24657041371'-001 9.53474742320'-001 6.99860980909'-001 1.42172543503'-001	10.124
0.090	-1.18567264661'-001 -1.42065027372'-001 -1.37633701561'-001 -1.43452959183'-001	9.50816207158'-001 4.23066762407'-001 6.97518967849'-001 1.41616121695'-001	10.209
0.100	-1.36804113812'-001 -1.15083089506'-001 -1.32761529476'-001 -1.38067155669'-001	4.21647286104'-001 9.48479526669'-001 6.95430902299'-001 1.41119510685'-001	10.288

Type I	N=8	$\psi_g = 1.0$	
δ	RE	$\pm IM$	$T_m \omega_g$
0.110	-1.11903072146'-001 -1.28357204547'-001 -1.32064711383'-001 -1.33220585496'-001	9.46399973222'-001 6.93548251584'-001 4.20366348470'-001 1.40671427678'-001	10.361
0.120	-1.08974710569'-001 -1.24338462556'-001 -1.27753806602'-001 -1.28816603774'-001	9.44530312070'-001 6.91835260484'-001 4.19200096399'-001 1.40263526107'-001	10.430
0.130	-1.06258063908'-001 -1.20642873022'-001 -1.23801006237'-001 -1.24782143852'-001	9.42835053993'-001 6.90264815701'-001 4.18130388731'-001 1.39889454888'-001	10.496
0.140	-1.03722028613'-001 -1.20151764888'-001 -1.17221934714'-001 -1.21060642552'-001	9.41286878569'-001 4.17143099490'-001 6.88815863105'-001 1.39544266647'-001	10.560
0.150	-1.16762969079'-001 -1.01341900136'-001 -1.17607430650'-001 -1.14037218358'-001	4.16227010290'-001 9.39864323296'-001 1.39224031002'-001 6.87471719637'-001	10.621
0.160	-9.90977230940'-002 -1.14386625399'-001 -1.13599957025'-001 -1.11057755876'-001	9.38550228704'-001 1.38925572566'-001 4.15373059663'-001 6.86218938718'-001	10.680
0.170	-9.69731408441'-002 -1.11368973780'-001 -1.10634449804'-001 -1.08258226956'-001	9.37330668519'-001 1.38646288033'-001 4.14573821123'-001 6.85046520375'-001	10.739
0.180	-9.49545751282'-002 -1.07843077276'-001 -1.08530318717'-001 -1.05617662426'-001	9.36194190137'-001 4.13823130504'-001 1.38384016587'-001 6.83945348719'-001	10.796
0.190	-9.30306259943'-002 -1.05850482967'-001 -1.05206306624'-001 -1.03118500923'-001	9.35131265844'-001 1.38136943784'-001 4.13115813390'-001 6.82907784329'-001	10.853
0.200	-9.11916257137'-002 -1.03312442989'-001 -1.02707646638'-001 -1.00745888292'-001	9.34133986702'-001 1.37903531544'-001 4.12447483017'-001 6.81927357582'-001	10.909

Type I	N = 8	$\psi_g = 1.0$	
δ	RE	$\pm IM$	$\Gamma_m \omega_g$
0.210	-8.94293031925'-002 -1.00901705268'-001 -1.00333048940'-001 -9.84871462206'-002	9.33195258465'-001 1.37682464010'-001 4.11814387575'-001 6.80998543074'-001	10.965
0.220	-9.80704480426'-002 -8.77365232882'-002 -9.86058298804'-002 -9.63313673965'-002	4.11213292253'-001 9.32309569660'-001 1.37472606832'-001 6.80116581556'-001	11.021
0.230	-8.61070867795'-002 -9.64140604883'-002 -9.59094056489'-002 -9.42690991360'-002	9.31471812444'-001 1.37272974971'-001 4.10641388138'-001 6.79277344607'-001	11.077
0.240	-8.45355710044'-002 -9.38408306392'-002 -9.43170336977'-002 -9.22920946068'-002	9.30677641234'-001 4.10096220468'-001 1.37082707293'-001 6.78477226757'-001	11.133
0.250	-8.30172033930'-002 -9.23065491734'-002 -9.18567558911'-002 -9.03931162393'-002	9.29923262371'-001 1.36901046844'-001 4.09575630500'-001 6.77713060839'-001	11.189
0.260	-8.15477601464'-002 -9.03753837229'-002 -8.99501613385'-002 -8.85657756007'-002	9.29205345110'-001 1.36727324603'-001 4.09077710443'-001 6.76982049190'-001	11.246
0.270	-8.01234835963'-002 -8.85171424197'-002 -8.81148280387'-002 -8.68044051967'-002	9.28520949747'-001 1.36560945539'-001 4.08600764484'-001 6.76281706663'-001	11.303
0.280	-7.87410145349'-002 -8.67261357525'-002 -8.63452207668'-002 -8.51039533081'-002	9.27867469294'-001 1.36401379456'-001 4.08143280078'-001 6.75609817721'-001	11.361
0.290	-7.73973359756'-002 -8.49972793913'-002 -8.46363908422'-002 -8.34598967376'-002	9.27242581114'-001 1.36248150877'-001 4.07703901044'-001 6.74964397324'-001	11.420
0.300	-7.60897276814'-002 -8.33260107975'-002 -8.29838949940'-002 -8.18681687670'-002	9.26644208259'-001 1.36100831511'-001 4.07281406762'-001 6.74343659419'-001	11.479

Type I	N=9	$\Psi_g = 1.0$	
δ	RE	$\pm IM$	$T_m \omega_g$
0.010	-2.28230266617'-001	2.65524827122'-001	10.285
	-1.61359947622'-001	1.00692315411'+000	
	-2.07707104986'-001	7.77530240477'-001	
	-2.22569556790'-001	5.26846080087'-001	
	-2.29769946353'-001		
0.020	-1.94766115965'-001	2.60319157409'-001	10.706
	-1.43709749278'-001	9.89783784040'-001	
	-1.80058087688'-001	7.64081803791'-001	
	-1.90785508654'-001	5.16991121374'-001	
	-1.95840618486'-001		
0.030	-1.75643528455'-001	2.57265143770'-001	10.962
	-1.33199621261'-001	9.80349015590'-001	
	-1.63908671937'-001	7.56343663186'-001	
	-1.72500491171'-001	5.11229410859'-001	
	-1.76488689831'-001		
0.040	-1.62284204755'-001	2.55092397598'-001	11.150
	-1.25623274453'-001	9.73918969452'-001	
	-1.52456718915'-001	7.50892551499'-001	
	-1.59670234192'-001	5.07135639931'-001	
	-1.62985407296'-001		
0.050	-1.49802245731'-001	5.03956318207'-001	11.300
	-1.19661329874'-001	9.69083432010'-001	
	-1.43576193157'-001	7.46681641616'-001	
	-1.52042880774'-001	2.53404111547'-001	
	-1.52642927671'-001		
0.060	-1.43754110708'-001	2.52023498029'-001	11.426
	-1.14724902342'-001	9.65232248916'-001	
	-1.36321140348'-001	7.43250926691'-001	
	-1.41794713913'-001	5.01356791698'-001	
	-1.44278190226'-001		
0.070	-1.36801477162'-001	2.50856110673'-001	11.535
	-1.10499107042'-001	9.62047127112'-001	
	-1.30187213916'-001	7.40357139322'-001	
	-1.35063345144'-001	4.99158653547'-001	
	-1.37265930509'-001		
0.080	-1.30819614446'-001	2.49845451021'-001	11.633
	-1.06795501059'-001	9.59341502068'-001	
	-1.24873424076'-001	7.37855904241'-001	
	-1.29261054623'-001	4.97255341892'-001	
	-1.31235767425'-001		

Type I	N=9	$\Psi_g = 1.0$	
δ	RE	$\pm 1M$	$T_m \omega_g$
0.090	-1.25574273152'-001 -1.03492215811'-001 -1.20185632954'-001 -1.24164895915'-001 -1.25950361227'-001	2.48954986608'-001 9.56996882906'-001 7.35654463511'-001 4.95578029200'-001	11.722
0.100	-1.20906356540'-001 -1.00505816128'-001 -1.15991276922'-001 -1.19623197068'-001 -1.21248589974'-001	2.48159706400'-001 9.54933440356'-001 7.33689648332'-001 4.94079635246'-001	11.805
0.110	-1.16702924701'-001 -9.77766327887'-002 -1.12195789385'-001 -1.15528130205'-001 -1.17016122909'-001	2.47441721236'-001 9.53094908597'-001 7.31916477162'-001 4.92726508499'-001	11.882
0.120	-9.52604866143'-002 -1.11800314837'-001 -1.08729335356'-001 -1.12880944631'-001 -1.13168935292'-001	9.51440180834'-001 4.91493744390'-001 7.30301769247'-001 2.46787785327'-001	11.955
0.130	-1.09377607452'-001 -9.29237376477'-002 -1.05538868840'-001 -1.08379631183'-001 -1.09643489250'-001	2.46187810506'-001 9.49938339651'-001 7.28820334334'-001 4.90362383192'-001	12.024
0.140	-1.06144267476'-001 -9.07401647128'-002 -1.02583146756'-001 -1.05219464324'-001 -1.06390589481'-001	2.45633933060'-001 9.48565562998'-001 7.27452591527'-001 4.89317650372'-001	12.091
0.150	-1.03142490367'-001 -8.86889238427'-002 -9.98294665151'-002 -1.02282945620'-001 -1.03371378464'-001	2.45119901265'-001 9.47303117835'-001 7.26183011229'-001 4.88347802311'-001	12.156
0.160	-8.67531648255'-002 -9.95404178102'-002 -9.72514563036'-002 -1.00341388774'-001 -1.00554636996'-001	9.46136015675'-001 4.87443348204'-001 7.24999064434'-001 2.44640661287'-001	12.220

Type I	N= 9	$\psi_g = 1.0$	
δ	RE	$\pm 1M$	$T_m \omega_g$
0.170	-8.49190666741'-002 -9.48275394330'-002 -9.77157723610'-002 -9.69676709392'-002 -9.79149101586'-002	9.45052082203'-001 7.23890495661'-001 2.44192068152'-001 4.86596503699'-001	12.282
0.180	-8.31751464684'-002 -9.25398372783'-002 -9.52448306307'-002 -9.45446876576'-002 -9.54311738642'-002	9.44041297582'-001 7.22848798017'-001 2.43770678615'-001 4.85800803661'-001	12.344
0.190	-8.15117582861'-002 -9.03733665132'-002 -9.29111756486'-002 -9.22547295704'-002 -9.30858655374'-002	9.43095320657'-001 7.21866837179'-001 2.43373601039'-001 4.85050818887'-001	12.404
0.200	-7.99207164346'-002 -8.83154471594'-002 -9.07001319174'-002 -9.00836576439'-002 -9.08641652386'-002	9.42207134965'-001 7.20938568957'-001 2.42998382437'-001 4.84341943389'-001	12.465
0.210	-7.83950129480'-002 -8.63552512055'-002 -8.85992011390'-002 -8.80194223209'-002 -8.87534541325'-002	9.41370784189'-001 7.20058827341'-001 2.42642924073'-001 4.83670236659'-001	12.525
0.220	-8.44834577656'-002 -7.69285995775'-002 -8.65976522255'-002 -8.60516706016'-002 -8.67428997553'-002	7.19223159133'-001 9.40581171053'-001 2.42305415421'-001 4.83032299795'-001	12.585
0.230	-8.26919853425'-002 -7.55162176283'-002 -8.46862035651'-002 -8.41714416856'-002 -8.48231344207'-002	7.18427700358'-001 9.39833899516'-001 2.41984283879'-001 4.82425179180'-001	12.646
0.240	-8.28567735880'-002 -7.41532647917'-002 -8.09737806558'-002 -8.23709280667'-002 -8.29860028534'-002	2.41678154871'-001 9.39125153527'-001 7.17669075511'-001 4.81846293755'-001	12.706

Type I	N= 9	$\psi_g = 10$	
δ	RE	$\pm IM$	$T_m \omega_g$
0.250	-7.28356883737'-002 -7.93226493588'-002 -8.11022817325'-002 -8.06432857098'-002 -8.12243613400'-002	9.39451599523'-001 7.16944315602'-001 2.41385818837'-001 4.81293371966'-001	12.768
0.260	-7.15599001087'-002 -7.77331229510'-002 -7.94164901708'-002 -7.89824810345'-002 -7.95319171593'-002	9.37810308977'-001 7.16250799524'-001 2.41106206755'-001 4.80764406035'-001	12.830
0.270	-7.03227053491'-002 -7.62003469936'-002 -7.77938742249'-002 -7.73831676052'-002 -7.79030980957'-002	9.37198696309'-001 7.15586195554'-001 2.40838368194'-001 4.80257610998'-001	12.892
0.280	-6.91212471138'-002 -7.47199929901'-002 -7.62295177592'-002 -7.58405849338'-002 -7.63329462789'-002	9.36614466664'-001 7.14948426146'-001 2.40581455058'-001 4.79771393355'-001	12.956
0.290	-6.79529582230'-002 -7.32881838776'-002 -7.47190262373'-002 -7.43504752172'-002 -7.48170305087'-002	9.36055575647'-001 7.14335626777'-001 2.40334706738'-001 4.79304324137'-001	13.020
0.300	-6.68155228377'-002 -7.19014317619'-002 -7.32584549369'-002 -7.29090149689'-002 -7.33513738087'-002	9.35520193278'-001 7.13746121168'-001 2.40097438727'-001 4.78855117955'-001	13.085

APPENDIX IV

Solution of the Equiripple Approximation of the differentiation of delay Time $V(\omega)$ "TYPE II"

1. Relations

$$V(\omega) = \sum_{i=1}^N \frac{2 \omega p_i}{(\omega^2 + p_i^2)^2}$$

In the approximation interval $V(\omega)$ has $(N-1)$ peaks at which

$$V(\omega) = \pm \delta \cdot T_m^2$$

2. Parameters

Mean delay time

$$T_m$$

Upper edge frequency

$$\omega_g$$

Delay-bandwidth product

$$T_m \omega_g$$

Normalized frequency

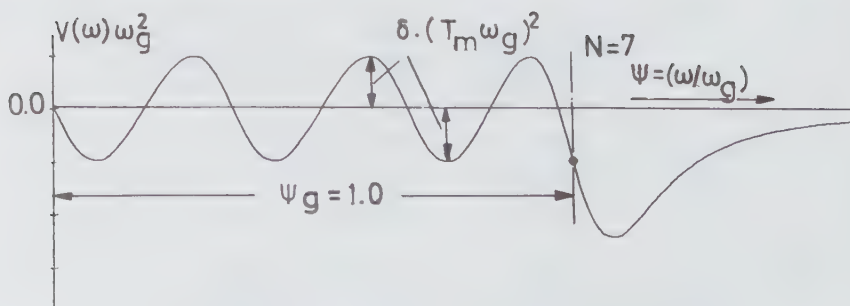
$$\psi = (\omega / \omega_g)$$

Normalized poles

$$RE \pm j IM = (p_i / \omega_g)$$

Normalized ripple magnitude

$$\delta \cdot (T_m \omega_g)^2$$



Frequency normalized $V(\omega)$

Type II	N = 2	$\psi_g = 1.0$	
δ	RE	$\pm 1M$	$T_m \omega_g$
0.020	-2.57502619445'+000	1.64825357311'+000	0.553
0.040	-2.00634329696'+000	1.36089210352'+000	0.688
0.060	-1.72555932734'+000	1.22714387073'+000	0.781
0.080	-1.54595730774'+000	1.14580123278'+000	0.852
0.100	-1.41665295803'+000	1.08987032337'+000	0.911
0.120	-1.31693806162'+000	1.04856378969'+000	0.962
0.140	-1.23648956168'+000	1.01659248228'+000	1.006
0.160	-1.16947123781'+000	9.91012176371'-001	1.046
0.180	-1.11228595907'+000	9.70035068567'-001	1.082
0.200	-1.06257094373'+000	9.52503466990'-001	1.115
0.220	-1.01869737159'+000	9.37630524753'-001	1.146
0.240	-9.79498003409'-001	9.24860104657'-001	1.175
0.260	-9.44109827229'-001	9.13786670786'-001	1.202
0.280	-9.11877920084'-001	9.04106725226'-001	1.228
0.300	-8.82294391471'-001	8.95588323161'-001	1.253
0.320	-8.54957635478'-001	8.88050694914'-001	1.276
0.340	-8.29544618016'-001	8.81350491971'-001	1.299
0.360	-8.05792574505'-001	8.75373393395'-001	1.321
0.380	-7.83483922045'-001	8.70026072946'-001	1.343
0.400	-7.62436901743'-001	8.65232108088'-001	1.364
0.420	-7.42498004740'-001	8.60928341848'-001	1.385
0.440	-7.23536273032'-001	8.57062187599'-001	1.406
0.460	-7.05439048677'-001	8.53589731905'-001	1.426
0.480	-6.88108085535'-001	8.50473580540'-001	1.447

Type II	N = 2	$\psi_g = 1.0$	
δ	RE	$\pm IM$	$T_m \omega_g$
0.500	-6.71457727272'-001	8.47682678614'-001	1.467
0.520	-6.55411889497'-001	8.45190293352'-001	1.488
0.540	-6.39902768566'-001	8.42973790706'-001	1.509
0.560	-6.24869722720'-001	8.41014449688'-001	1.530
0.580	-6.10257651235'-001	8.39296429767'-001	1.551
0.600	-5.96015758532'-001	8.37806039872'-001	1.573

Type II	N = 3	$\psi_g = 1.0$	
δ	RE	$\pm 1M$	$T_m \omega_g$
0.020	-9.86547316930' -001 -1.19298405061' +000	1.26160591526' +000	1.598
0.040	-8.23907957223' -001 -9.79142770637' -001	1.15046738822' +000	1.820
0.060	-7.36512904230' -001 -8.63844787858' -001	1.09549758781' +000	1.962
0.080	-6.77518032113' -001 -7.86052926327' -001	1.06073528953' +000	2.069
0.100	-6.33297808090' -001 -7.27888011941' -001	1.03612024081' +000	2.155
0.120	-5.98067025792' -001 -6.81718224805' -001	1.01750498766' +000	2.227
0.140	-5.68847661481' -001 -6.43597872695' -001	1.00280652632' +000	2.290
0.160	-5.43912020006' -001 -6.11229504779' -001	9.90842623680' -001	2.347
0.180	-5.22171478686' -001 -5.83161082788' -001	9.80882919677' -001	2.398
0.200	-5.02897421280' -001 -5.58418260596' -001	9.72447629309' -001	2.445
0.220	-4.73407843452' -001 -5.20272821188' -001	9.66195113117' -001	2.510
0.240	-4.69844231961' -001 -5.16356430278' -001	9.58922010166' -001	2.529
0.260	-4.55414687898' -001 -4.98163906785' -001	9.53420909271' -001	2.569
0.280	-4.42076042786' -001 -4.81450605512' -001	9.48571089171' -001	2.606
0.300	-4.29660045698' -001 -4.65990097568' -001	9.44270813056' -001	2.642
0.320	-4.18032140937' -001 -4.51601284542' -001	9.40440165359' -001	2.677

Type II	N = 3	$\psi_g = 1.0$	
δ	RE	$\pm 1M$	$T_m \omega_g$
0.340	-4.07082970654'-001 -4.38137032645'-001	9.37015398111'-001	2.712
0.360	-3.96722384467'-001 -4.25476143995'-001	9.33945004595'-001	2.746
0.380	-3.86875120988'-001 -4.13517558802'-001	9.31186938138'-001	2.779
0.400	-3.77477559694'-001 -4.02176006872'-001	9.28706429037'-001	2.813
0.420	-3.68475405419'-001 -3.91378872562'-001	9.26474655367'-001	2.846
0.440	-3.59821863349'-001 -3.81063738372'-001	9.24467616493'-001	2.880
0.460	-3.51476102835'-001 -3.71176352172'-001	9.22664952788'-001	2.914
0.480	-3.43402284834'-001 -3.61669279766'-001	9.21049622237'-001	2.949
0.500	-3.35568573231'-001 -3.52500606343'-001	9.19607107200'-001	2.984
0.520	-3.27946538881'-001 -3.43633103039'-001	9.18325317467'-001	3.020
0.540	-3.20510433041'-001 -3.35033296244'-001	9.17193876539'-001	3.057
0.560	-3.13236774320'-001 -3.26670898110'-001	9.16204047247'-001	3.096
0.580	-3.06104050127'-001 -3.18518380911'-001	9.15348850533'-001	3.135
0.600	-2.99092171674'-001 -3.10550314469'-001	9.14622337728'-001	3.177

Type II	N = 4	$\psi_g = 1.0$	
δ	RE	$\pm 1M$	$T_m \omega_g$
0.020	-5.79984062537'-001 -7.29079302291'-001	1.15263485420'+000 3.97233314965'-001	2.831
0.040	-4.98017827878'-001 -6.10243747622'-001	1.08901474474'+000 3.77678151592'-001	3.109
0.060	-4.52283941857'-001 -5.44247950282'-001	1.05656019022'+000 3.67482653528'-001	3.283
0.080	-4.98921632428'-001 -4.20640992161'-001	3.60739443619'-001 1.03561602434'+000	3.412
0.100	-3.96470517251'-001 -4.64592694814'-001	1.02055659465'+000 3.55769747737'-001	3.515
0.120	-3.76911724583'-001 -4.37064600403'-001	1.009026111582'+000 3.51872183110'-001	3.602
0.140	-3.60471974054'-001 -4.14141836445'-001	9.99827094584'-001 3.48690063722'-001	3.678
0.160	-3.46275601311'-001 -3.94533873157'-001	9.92272728214'-001 3.46018333206'-001	3.746
0.180	-3.33765731899'-001 -3.77418863223'-001	9.85935186094'-001 3.43728779790'-001	3.808
0.200	-3.22566509301'-001 -3.62241638402'-001	9.80531080681'-001 3.41736061324'-001	3.865
0.220	-3.12412453903'-001 -3.48609898734'-001	9.75863861906'-001 3.39980650842'-001	3.919
0.240	-3.03109085617'-001 -3.36236277275'-001	9.71791971228'-001 3.38419398209'-001	3.969
0.260	-2.94509705491'-001 -3.24904096589'-001	9.68210517192'-001 3.37020112875'-001	4.018
0.280	-2.86500868482'-001 -3.14445941487'-001	9.65039777726'-001 3.35758148744'-001	4.066
0.300	-2.78992953787'-001 -3.04729739291'-001	9.62217792847'-001 3.34614200740'-001	4.112
0.320	-2.71913970249'-001 -2.95649553955'-001	9.59695917136'-001 3.33573002023'-001	4.157

Type II	N = 4	$\psi_g = 1.0$	
δ	RE	$\pm 1M$	$T_m \psi_g$
0.340	-2.65205002197'-001 -2.87118917902'-001	9.57434852749'-001 3.32622109629'-001	4.202
0.360	-2.58817322087'-001 -2.79066475083'-001	9.55403055912'-001 3.31751464717'-001	4.247
0.380	-2.52709986733'-001 -2.71432480102'-001	9.53574516101'-001 3.30952700610'-001	4.291
0.400	-2.46848215491'-001 -2.64166380228'-001	9.51927674763'-001 3.30218824878'-001	4.337
0.420	-2.41202186327'-001 -2.57225010930'-001	9.50444702976'-001 3.29544010267'-001	4.382
0.440	-2.35746019749'-001 -2.50571114906'-001	9.49110604946'-001 3.28923314257'-001	4.429
0.460	-2.30457046757'-001 -2.44172253697'-001	9.47912741311'-001 3.28352536581'-001	4.476
0.480	-2.25315222961'-001 -2.37999942070'-001	9.46840431196'-001 3.27828098725'-001	4.525
0.500	-2.20302668622'-001 -2.32028966497'-001	9.45884677120'-001 3.27346961592'-001	4.575
0.520	-2.15403290022'-001 -2.26236828592'-001	9.45037929578'-001 3.26906553826'-001	4.627
0.540	-2.10602425683'-001 -2.20603245722'-001	9.44293712397'-001 3.26504650177'-001	4.681
0.560	-2.05886644984'-001 -2.15109835601'-001	9.43646724434'-001 3.26139412646'-001	4.737
0.580	-2.01243471529'-001 -2.09739740836'-001	9.43092470672'-001 3.25809267404'-001	4.795
0.600	-1.96661236645'-001 -2.04477405117'-001	9.42627362554'-001 3.25512943553'-001	4.856

Type II	N = 5	$\psi_g = 1.0$	
δ	RE	$\pm 1M$	$T_m \omega_g$
0.020	-4.03527032500'-001 -5.13224322465'-001 -5.33956011350'-001	1.10514644516'+000 5.74052930082'-001	4.155
0.040	-3.51866978428'-001 -4.34912394579'-001 -4.48981649563'-001	1.06181519688'+000 5.53612578311'-001	4.473
0.060	-3.22386655436'-001 -3.90653144652'-001 -4.01367483777'-001	1.03928252915'+000 5.42580666312'-001	4.670
0.080	-3.01681110388'-001 -3.59912289616'-001 -3.68513791094'-001	1.02455853432'+000 5.35120743298'-001	4.816
0.100	-2.85680548263'-001 -3.36431699296'-001 -3.43554726118'-001	1.01387166511'+000 5.29534051064'-001	4.932
0.120	-2.72607722054'-001 -3.17472147330'-001 -3.23493819611'-001	1.00562729163'+000 5.25098446098'-001	5.030
0.140	-2.61528004030'-001 -3.01590953030'-001 -3.06757207116'-001	9.99008574338'-001 5.21441403776'-001	5.116
0.160	-2.51889553154'-001 -2.87935523927'-001 -2.92417151490'-001	9.93544060366'-001 5.18346174496'-001	5.193
0.180	-2.43339482669'-001 -2.75960407831'-001 -2.79881561917'-001	9.88938446641'-001 5.15675801755'-001	5.264
0.200	-2.35638472117'-001 -2.65295694469'-001 -2.68749821953'-001	9.849959228751'-001 5.13338300323'-001	5.330
0.220	-2.34649864651'-001 -2.64182661922'-001 -2.67622297797'-001	9.80862738928'-001 5.11184619500'-001	5.352
0.240	-2.22149369558'-001 -2.46917699333'-001 -2.49639545778'-001	9.78586200046'-001 5.09420504935'-001	5.451

Type II	N = 5	$\psi_g = 1.0$	
δ	RE	$\pm 1M$	$T_m \omega_g$
0.260	-2.16141854991'-001 -2.38865868159'-001 -2.41296244079'-001	9.75947819126'-001 5.07757485961'-001	5.509
0.280	-2.10520803357'-001 -2.31410605345'-001 -2.33587217037'-001	9.73606195701'-001 5.06252555279'-001	5.565
0.300	-2.05228076051'-001 -2.24462606926'-001 -2.26416722660'-001	9.71517674770'-001 5.04884236005'-001	5.620
0.320	-2.00216755064'-001 -2.17949995811'-001 -2.19707844197'-001	9.69647642720'-001 5.03635324934'-001	5.675
0.340	-1.95448408045'-001 -2.11814065726'-001 -2.13397870102'-001	9.67968420278'-001 5.02491928190'-001	5.730
0.360	-1.90891056467'-001 -2.06006158573'-001 -2.07434915967'-001	9.66457364815'-001 5.01442563142'-001	5.785
0.380	-1.86517823964'-001 -2.00485530615'-001 -2.01775609035'-001	9.65096123917'-001 5.00477831662'-001	5.841
0.400	-1.82305748007'-001 -1.95217569308'-001 -1.96383171311'-001	9.63869159645'-001 4.99589695864'-001	5.898
0.420	-1.78235128689'-001 -1.90172713192'-001 -1.91226241557'-001	9.62763899250'-001 4.98771586728'-001	5.955
0.440	-1.74288757287'-001 -1.85325331568'-001 -1.86277679946'-001	9.61769450205'-001 4.98017757767'-001	6.014
0.460	-1.70451544593'-001 -1.80653099276'-001 -1.81513886579'-001	9.60876805006'-001 4.97323407810'-001	6.075
0.480	-1.66710108530'-001 -1.76136376274'-001 -1.76914137242'-001	9.60078392054'-001 4.96684457692'-001	6.138

Type II	N = 5	$\psi_g = 1.0$	
δ	RE	$\pm 1M$	$T_m \omega_g$
0.500	-1.63052437969'-001 -1.71757707445'-001 -1.72460048452'-001	9.59367671192'-001 4.96097350460'-001	6.203
0.520	-1.59467694553'-001 -1.67501496046'-001 -1.68135224281'-001	9.58739236143'-001 4.95559108851'-001	6.271
0.540	-1.63353659562'-001 -1.55945975494'-001 -1.63924890258'-001	4.95067171170'-001 9.58188488061'-001	6.342
0.560	-1.52478152864'-001 -1.59301383475'-001 -1.59815629828'-001	9.57711581785'-001 4.94619367164'-001	6.416
0.580	-1.49055740068'-001 -1.55332917333'-001 -1.55795166476'-001	9.57305413136'-001 4.94213912971'-001	6.493
0.600	-1.45670737653'-001 -1.51437360953'-001 -1.51852138108'-001	9.56967373331'-001 4.93849286082'-001	6.575

Type II	N = 6	$\psi_g = 1.0$	
δ	RE	$\pm 1M$	$T_m \omega_g$
0.020	-3.06914552679'-001 -3.92150951077'-001 -4.13163081190'-001	1.07933736860'+000 6.72948420892'-001 2.28270886757'-001	5.533
0.040	-3.35170902239'-001 -2.70280979446'-001 -3.49399365980'-001	6.54097123050'-001 1.04688997380'+000 2.21650569292'-001	5.882
0.060	-3.02563194222'-001 -2.49050865328'-001 -3.13391041247'-001	6.43705100381'-001 1.02979513101'+000 2.17918204697'-001	6.097
0.080	-2.33984630231'-001 -2.79730195856'-001 -2.88420622041'-001	1.01852947002'+000 6.36586140172'-001 2.15327480270'-001	6.255
0.100	-2.62181714294'-001 -2.22247949427'-001 -2.69377809756'-001	6.31205920479'-001 1.01030079200'+000 2.13352404357'-001	6.382
0.120	-2.47940247613'-001 -2.12594542070'-001 -2.54023574307'-001	6.26904708410'-001 1.00392061041'+000 2.11763721443'-001	6.490
0.140	-2.35959373353'-001 -2.04365523066'-001 -2.41178462282'-001	6.23339070040'-001 9.98776869135'-001 2.10440771629'-001	6.584
0.160	-2.25618336470'-001 -1.97170204672'-001 -2.30145656427'-001	6.20307804387'-001 9.94514939583'-001 2.09312212704'-001	6.669
0.180	-2.16518668101'-001 -1.90757795062'-001 -2.20479542160'-001	6.17682902510'-001 9.90911744939'-001 2.08332300063'-001	6.747
0.200	-2.08389385676'-001 -1.84957675148'-001 -2.11878127710'-001	6.15377985483'-001 9.87818458852'-001 2.07469956182'-001	6.821
0.220	-2.01037785275'-001 -1.79648403708'-001 -2.04127361209'-001	6.13331888209'-001 9.85130911209'-001 2.06703048909'-001	6.891
0.240	-1.94321985837'-001 -1.74740470218'-001 -1.97070123948'-001	6.11499919633'-001 9.82773843469'-001 2.06015320528'-001	6.958

Type II	N=6	$\psi_g = 1.0$	
δ	RE	$\pm 1M$	$T_m \omega_g$
0.260	-1.88134424552'-001 -1.70165897594'-001 -1.90587651083'-001	6.09848291431'-001 9.80691008313'-001 2.05394425910'-001	7.024
0.280	-1.82391601851'-001 -1.65871806083'-001 -1.84588003955'-001	6.08350891769'-001 9.78839424344'-001 2.04830786024'-001	7.089
0.300	-1.77027363654'-001 -1.61816191852'-001 -1.78998538078'-001	6.06987106254'-001 9.77185538963'-001 2.04316812491'-001	7.153
0.320	-1.71988406517'-001 -1.57965114253'-001 -1.73760869569'-001	6.05740491890'-001 9.75702882977'-001 2.03846429682'-001	7.217
0.340	-1.67231082549'-001 -1.54290676059'-001 -1.68827311805'-001	6.04597617010'-001 9.74370069882'-001 2.03414661715'-001	7.281
0.360	-1.62719207245'-001 -1.50769661821'-001 -1.64158427266'-001	6.03547530664'-001 9.73169854224'-001 2.03017436670'-001	7.347
0.380	-1.58422348923'-001 -1.47382441006'-001 -1.59721132019'-001	6.02580967276'-001 9.72087793532'-001 2.02651305826'-001	7.413
0.400	-1.54314699539'-001 -1.44112281942'-001 -1.55487433109'-001	6.01690269701'-001 9.71112086569'-001 2.02313407637'-001	7.481
0.420	-1.50374119137'-001 -1.40944741110'-001 -1.51433373976'-001	6.00868967609'-001 9.70232867418'-001 2.02001317637'-001	7.551
0.440	-1.46581436758'-001 -1.37867222336'-001 -1.47538259786'-001	6.00111574175'-001 9.69441860256'-001 2.01712974184'-001	7.623
0.460	-1.42919889666'-001 -1.34868620342'-001 -1.43784040141'-001	5.99413366675'-001 9.68732015695'-001 2.01446599065'-001	7.697
0.480	-1.39374704453'-001 -1.31939058905'-001 -1.40154845791'-001	5.98770307668'-001 9.68097374560'-001 2.01200667463'-001	7.774

Type II	N = 6	$\psi_g = 1.0$	
δ	RE	$\pm 1M$	$T_m \omega_g$
0.500	-1.35932775094'-001	5.98179047654'-001	7.854
	-1.29069693852'-001	9.67533068077'-001	
	-1.36636631971'-001	2.00973904808'-001	
0.520	-1.32582325470'-001	5.97636599494'-001	7.938
	-1.26252480230'-001	9.67034789443'-001	
	-1.33216813905'-001	2.00765175116'-001	
0.540	-1.29312728524'-001	5.97140481401'-001	8.026
	-1.23480070230'-001	9.66599023210'-001	
	-1.29884064059'-001	2.00573526621'-001	
0.560	-1.26114286497'-001	5.96688537233'-001	8.118
	-1.20745661534'-001	9.66222753454'-001	
	-1.26628074599'-001	2.00398129269'-001	
0.580	-1.22978091662'-001	5.96278995017'-001	8.215
	-1.18042912796'-001	9.65903558779'-001	
	-1.23439404557'-001	2.00238293161'-001	
0.600	-1.19895875523'-001	5.95910371780'-001	8.317
	-1.15365837930'-001	9.65639458911'-001	
	-1.20309316822'-001	2.00093434778'-001	

Type II	N = 7	$\psi_g = 1.0$	
δ	RE	$\pm IM$	$T_m \omega_g$
0.020	-2.46566672264'-001 -3.34399160445'-001 -3.15720773291'-001 -3.38715872749'-001	1.06334798486'+000 3.74789278249'-001 7.35618310060'-001	6.945
0.040	-2.18647496131'-001 -2.84212573410'-001 -2.71540881415'-001 -2.87053741682'-001	1.03758160595'+000 3.65582153965'-001 7.18596569783'-001	7.319
0.060	-2.02282230668'-001 -2.55674495800'-001 -2.46018097332'-001 -2.57803884408'-001	1.02387734276'+000 3.60316657152'-001 7.09080612054'-001	7.549
0.080	-1.90578865877'-001 -2.35794053428'-001 -2.28035496533'-001 -2.37486050770'-001	1.01479062106'+000 3.56632802042'-001 7.02506592876'-001	7.718
0.100	-1.81407275373'-001 -2.20579776000'-001 -2.14149767402'-001 -2.21970649276'-001	1.00812314916'+000 3.53809715183'-001 6.97508927988'-001	7.854
0.120	-2.02837389712'-001 -1.73826086135'-001 -2.08276899077'-001 -2.09446056499'-001	6.93495759762'-001 1.00293451099'+000 3.51530314601'-001	7.969
0.140	-1.67335734361'-001 -1.93289352712'-001 -1.97958740060'-001 -1.98957193967'-001	9.98738785202'-001 6.90157297548'-001 3.49626699317'-001	8.071
0.160	-1.61635813365'-001 -1.89072863803'-001 -1.85020613895'-001 -1.89935609793'-001	9.95233565107'-001 3.47992098432'-001 6.87297364930'-001	8.164
0.180	-1.56544712871'-001 -1.81279092907'-001 -1.77732497701'-001 -1.82031379940'-001	9.92300198705'-001 3.46583065882'-001 6.84840533650'-001	8.249
0.200	-1.51922315679'-001 -1.74327537676'-001 -1.71202828613'-001 -1.74988183368'-001	9.89759978897'-001 3.45334970185'-001 6.82666758596'-001	8.330

Type II	N = 7	$\psi_g = 1.0$	
δ	RE	$\pm 1M$	$T_m \omega_g$
0.220	-1.47678812933'-001 -1.68052647550'-001 -1.65284898154'-001 -1.68636132337'-001	9.87549250182'-001 3.44223489174'-001 6.80733746016'-001	8.407
0.240	-1.43745466161'-001 -1.62329944607'-001 -1.59867743787'-001 -1.62847663880'-001	9.85607402407'-001 3.43225538993'-001 6.79000347194'-001	8.482
0.260	-1.40069999259'-001 -1.57065254789'-001 -1.54867133046'-001 -1.57526350398'-001	9.83889195450'-001 3.42323602858'-001 6.77435474601'-001	8.555
0.280	-1.36611670416'-001 -1.52185591436'-001 -1.50217573289'-001 -1.52597517161'-001	9.82359976577'-001 3.41504052085'-001 6.76015030535'-001	8.628
0.300	-1.33338069962'-001 -1.47633234723'-001 -1.45867111565'-001 -1.48002152419'-001	9.80992706700'-001 3.40756090103'-001 6.74719990013'-001	8.701
0.320	-1.30222944623'-001 -1.43361725674'-001 -1.41773806321'-001 -1.43692790327'-001	9.79765884607'-001 3.40071006827'-001 6.73535049322'-001	8.774
0.340	-1.27244704261'-001 -1.39333109240'-001 -1.37903295268'-001 -1.39630678583'-001	9.78662275575'-001 3.39441716358'-001 6.72447799676'-001	8.848
0.360	-1.24385329942'-001 -1.35515941220'-001 -1.34227030179'-001 -1.35783732956'-001	9.77667810104'-001 3.38862354503'-001 6.71447996971'-001	8.923
0.380	-1.21629582845'-001 -1.31883845417'-001 -1.30720997524'-001 -1.32125057055'-001	9.76770826372'-001 3.38327998452'-001 6.70527057635'-001	9.000
0.400	-1.18964452693'-001 -1.28414483515'-001 -1.27364806227'-001 -1.28631885251'-001	9.75961808945'-001 3.37834564016'-001 6.69677892001'-001	9.080

Type II	N = 7	$\psi_g = 1.0$	
δ	RE	$\pm 1M$	$T_m \omega_g$
0.420	-1.16378670267'-001 -1.25088700948'-001 -1.24140922093'-001 -1.25284708560'-001	9.75232637393'-001 3.37378531502'-001 6.68894391583'-001	9.161
0.440	-1.13862409147'-001 -1.21889955986'-001 -1.21034163123'-001 -1.22066688531'-001	9.74576632823'-001 3.36956950821'-001 6.68171463985'-001	9.246
0.460	-1.11406988272'-001 -1.18803806061'-001 -1.18031237114'-001 -1.18963132503'-001	9.73988076123'-001 3.36567267885'-001 6.67504705907'-001	9.333
0.480	-1.15120418341'-001 -1.09004677895'-001 -1.15817546626'-001 -1.15961123302'-001	6.66890368637'-001 9.73462162394'-001 3.36207299326'-001	9.425
0.500	-1.06648529709'-001 -1.12919909669'-001 -1.12291278292'-001 -1.13049211224'-001	9.72994874759'-001 3.35875183184'-001 6.66325272774'-001	9.520
0.520	-1.10100788999'-001 -1.04332210540'-001 -1.09534434560'-001 -1.10217133664'-001	3.35569245155'-001 9.72582612765'-001 6.65806552395'-001	9.620
0.540	-1.02049944013'-001 -1.07351100207'-001 -1.06841430163'-001 -1.07455671016'-001	9.72222620974'-001 3.35288141974'-001 6.65331948912'-001	9.725
0.560	-1.04662567286'-001 -9.97963706977'-002 -1.04204535197'-001 -1.04756429533'-001	3.35030693605'-001 9.71912520457'-001 6.64899482788'-001	9.835
0.580	-1.02027578498'-001 -9.75664595499'-002 -1.01616616612'-001 -1.02111694760'-001	3.34795843977'-001 9.71650193707'-001 6.64507383644'-001	9.952
0.600	-9.94391110580'-002 -9.53554791413'-002 -9.90710720434'-002 -9.95143536602'-002	3.34582779444'-001 9.71434150415'-001 6.64154328661'-001	10.075

APPENDIX V

Solution of the Equiripple Approximation of the phase function

B(ω) "TYPE III"

1. Relations

$$B(\omega) = \sum_{i=1}^N \arctan\left(\frac{\operatorname{Im}(P_i) - \omega}{\operatorname{Re}(P_i)}\right)$$

We define $\phi(\omega) = B(\omega) - T_m \omega$

In the approximation interval $\phi(\omega)$ has N peaks at which

$$\phi(\omega) = \pm \delta$$

2. Parameters

Mean delay time

$$T_m$$

Upper edge frequency

$$\omega_g$$

Delay-bandwidth product

$$T_m \omega_g$$

Normalized frequency

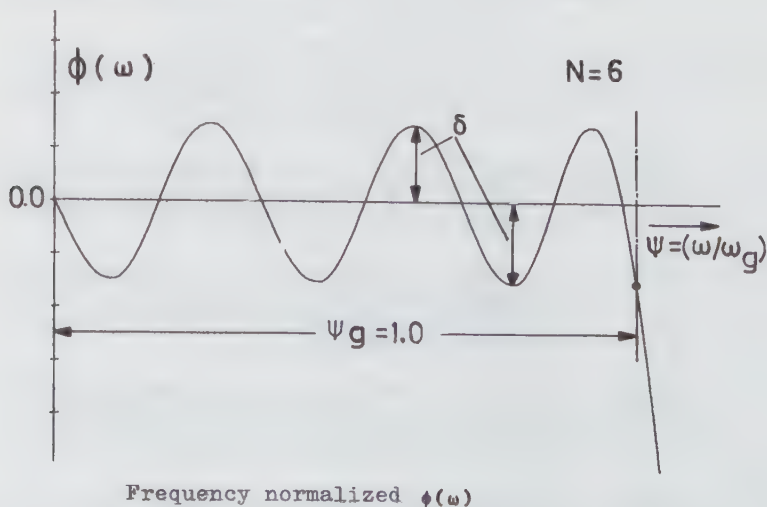
$$\psi = (\omega / \omega_g)$$

Normalized poles

$$RE \pm j \operatorname{Im} = (P_i / \omega_g)$$

Normalized ripple magnitude

$$\delta$$



Type III	N=2	$\psi_g = 1.0$	
δ	RE	$\pm IM$	$T_m \omega_g$
0.002	-1.26131994557'+000	8.81196367773'-001	1.076
0.004	-1.06144743739'+000	7.87638366717'-001	1.236
0.006	-9.54351448847'-001	7.39833706597'-001	1.341
0.008	-8.82283521165'-001	7.08755829368'-001	1.421
0.010	-8.28454867238'-001	6.86178869404'-001	1.486
0.012	-7.85738878345'-001	6.68677726360'-001	1.542
0.014	-7.46927039254'-001	6.51429623598'-001	1.598
0.016	-7.17481397584'-001	6.40001253989'-001	1.641
0.018	-6.91894284508'-001	6.30218954862'-001	1.680
0.020	-6.69307188884'-001	6.21701369721'-001	1.715
0.022	-6.49115850742'-001	6.14182544039'-001	1.748
0.024	-6.30879497887'-001	6.07470165578'-001	1.779
0.026	-6.14267127767'-001	6.01421009476'-001	1.807
0.028	-5.99024096209'-001	5.95925760757'-001	1.834
0.030	-5.84950471210'-001	5.90899240899'-001	1.860
0.032	-5.71886517279'-001	5.86273890473'-001	1.884
0.034	-5.59702675342'-001	5.81995296041'-001	1.907
0.036	-5.48292468018'-001	5.78019039902'-001	1.930
0.038	-5.37567365089'-001	5.74308431744'-001	1.951
0.040	-5.27452995288'-001	5.70832843809'-001	1.971
0.042	-5.17886302441'-001	5.67566467042'-001	1.991
0.044	-5.08813378616'-001	5.64487367839'-001	2.010
0.046	-5.00187790316'-001	5.61576762852'-001	2.028
0.048	-4.91969267496'-001	5.58818453599'-001	2.045
0.050	-4.84120401121'-001	5.56197388498'-001	2.062

Type III	N=2	$\psi_g = 1.0$	
δ	RE	$\pm 1M$	$T_m \omega_g$
0.052	-4.78435665576'-001	5.54772466836'-001	2.075
0.054	-4.71169837667'-001	5.52354889533'-001	2.091
0.056	-4.64199388989'-001	5.50045952259'-001	2.107
0.058	-4.57502226065'-001	5.47836916780'-001	2.122
0.060	-4.51058611398'-001	5.45720016198'-001	2.137
0.062	-4.44850842916'-001	5.43688318921'-001	2.152
0.064	-4.38862985266'-001	5.41735615345'-001	2.166
0.066	-4.33080644289'-001	5.39856322866'-001	2.180
0.068	-4.27490776117'-001	5.38045406283'-001	2.193
0.070	-4.22081524554'-001	5.36298309526'-001	2.206
0.072	-4.16842082326'-001	5.34610898198'-001	2.219
0.074	-4.11762571890'-001	5.32979410240'-001	2.232
0.076	-4.06833942408'-001	5.31400413162'-001	2.244
0.078	-4.02047881016'-001	5.29870767867'-001	2.256
0.080	-3.97396735109'-001	5.28387596518'-001	2.268
0.082	-3.92873444744'-001	5.26948255021'-001	2.280
0.084	-3.88471483679'-001	5.25550309165'-001	2.291
0.086	-3.84184806935'-001	5.24191513032'-001	2.303
0.088	-3.80007805336'-001	5.22869790835'-001	2.314
0.090	-3.75935264747'-001	5.21583220419'-001	2.325
0.092	-3.71962330086'-001	5.20330018801'-001	2.335
0.094	-3.68084473466'-001	5.19108529370'-001	2.346
0.096	-3.64297465560'-001	5.17917210440'-001	2.356
0.098	-3.60597350027'-001	5.16754625030'-001	2.366
0.100	-3.56980421002'-001	5.15619432183'-001	2.376

Type III	N=3	$\Psi_g = 1.0$	
δ	RE	$\pm 1M$	$T_m \omega_g$
0.002	-7.27408785340'-001 -8.79981791848'-001	9.73023264025'-001	2.107
0.004	-6.22356417844'-001 -7.45982573979'-001	9.02017558066'-001	2.346
0.006	-5.66245830974'-001 -6.74276757200'-001	8.66285951182'-001	2.493
0.008	-5.28177760654'-001 -6.25644930871'-001	8.42976254106'-001	2.602
0.010	-4.99493039104'-001 -5.89039835148'-001	8.25928581631'-001	2.689
0.012	-4.76543959929'-001 -5.59795474779'-001	8.12615832845'-001	2.762
0.014	-4.53870985402'-001 -5.31859195064'-001	8.00907483677'-001	2.835
0.016	-4.38009668200'-001 -5.11658619100'-001	7.91981437149'-001	2.890
0.018	-4.24126405102'-001 -4.94010614930'-001	7.84290605621'-001	2.939
0.020	-4.11791304319'-001 -4.78360131267'-001	7.77552129290'-001	2.984
0.022	-4.00699924670'-001 -4.64313993070'-001	7.71568339296'-001	3.026
0.024	-3.90628954116'-001 -4.51583752321'-001	7.66195989410'-001	3.064
0.026	-3.81409836798'-001 -4.39951666529'-001	7.61328157278'-001	3.100
0.028	-3.72912290713'-001 -4.29249427136'-001	7.56883050741'-001	3.134
0.030	-3.65033579474'-001 -4.19344310591'-001	7.52796792087'-001	3.166
0.032	-3.57691289148'-001 -4.10129850818'-001	7.49018602892'-001	3.196
0.034	-3.50818320934'-001 -4.01519372041'-001	7.45507493586'-001	3.224

Type III	N = 3	$\psi_g = 1.0$	
δ	RE	$\pm 1M$	$T_m \omega_g$
0.036	-3.44359333609'-001 -3.93441392720'-001	7.42229931111'-001	3.251
0.038	-3.38268158990'-001 -3.85836286470'-001	7.39158155760'-001	3.277
0.040	-3.32505889550'-001 -3.78653810796'-001	7.36268945129'-001	3.302
0.042	-3.27039438114'-001 -3.71851245538'-001	7.33542689603'-001	3.326
0.044	-3.21840436805'-001 -3.65391969943'-001	7.30962689700'-001	3.349
0.046	-3.16884384069'-001 -3.59244360188'-001	7.28514614995'-001	3.371
0.048	-3.12149975638'-001 -3.53380924425'-001	7.26186082899'-001	3.392
0.050	-3.07618573799'-001 -3.47777616662'-001	7.23966325735'-001	3.413
0.052	-3.03273781901'-001 -3.42413286970'-001	7.21845925429'-001	3.433
0.054	-2.991011100522'-001 -3.37269236778'-001	7.19816600493'-001	3.452
0.056	-2.95087646780'-001 -3.32328856181'-001	7.17871032681'-001	3.471
0.058	-2.91221923711'-001 -3.27577325610'-001	7.16002725414'-001	3.489
0.060	-2.87493629301'-001 -3.23001369063'-001	7.14205887340'-001	3.507
0.062	-2.83893496726'-001 -3.18589047718'-001	7.12475334807'-001	3.524
0.064	-2.80413161090'-001 -3.14329587505'-001	7.10806411298'-001	3.541
0.066	-2.77045046492'-001 -3.10213233111'-001	7.09194919575'-001	3.557

Type III	N=3	$\Psi_g = 1.0$	
δ	RE	$\pm IM$	$T_m \omega_g$
0.068	-2.73782269906'-001 -3.06231123963'-001	7.07637063653'-001	3.573
0.070	-2.70618559956'-001 -3.02375189029'-001	7.06129400420'-001	3.588
0.072	-2.67548186563'-001 -2.98638055894'-001	7.04668797749'-001	3.604
0.074	-2.64565900630'-001 -2.95012973082'-001	7.03252398678'-001	3.619
0.076	-2.61666882019'-001 -2.91493742501'-001	7.01877590622'-001	3.633
0.078	-2.58846694182'-001 -2.88074660974'-001	7.00541978913'-001	3.647
0.080	-2.56101244569'-001 -2.84750469138'-001	6.99243363284'-001	3.661
0.082	-2.53426750194'-001 -2.81516306968'-001	6.97979717817'-001	3.675
0.084	-2.50819707362'-001 -2.78367674408'-001	6.96749173578'-001	3.688
0.086	-2.48276864721'-001 -2.75300396977'-001	6.95550002522'-001	3.701
0.088	-2.45795199950'-001 -2.72310595305'-001	6.94380604506'-001	3.714
0.090	-2.43371898632'-001 -2.69394658040'-001	6.93239494671'-001	3.727
0.092	-2.41004335698'-001 -2.66549217900'-001	6.92125293081'-001	3.740
0.094	-2.38690059028'-001 -2.63771130340'-001	6.91036715241'-001	3.752
0.096	-2.36426774442'-001 -2.61057454321'-001	6.89972563574'-001	3.764
0.098	-2.34212332587'-001 -2.58405435357'-001	6.88931719731'-001	3.776
0.100	-2.32044717031'-001 -2.55812490097'-001	6.87913138136'-001	3.787

Type III	N = 4	$\psi_g = 1.0$	
δ	RE	$\pm IM$	$T_m \omega_g$
0.002	-4.73927448289'-001 -6.05316384226'-001	9.77774379571'-001 3.31683938832'-001	3.365
0.004	-4.24365317645'-001 -5.36474815479'-001	9.34808639275'-001 3.18849684019'-001	3.601
0.006	-3.89430895611'-001 -4.87127570137'-001	9.09287310795'-001 3.10757945172'-001	3.776
0.008	-3.65694948006'-001 -4.53721396290'-001	8.92523543287'-001 3.05409003577'-001	3.903
0.010	-3.47606954783'-001 -4.28410208289'-001	8.80059766769'-001 3.01386024174'-001	4.005
0.012	-3.33009829894'-001 -4.08087502838'-001	8.70204181323'-001 2.98172754323'-001	4.090
0.014	-3.20782029507'-001 -3.91142161934'-001	8.62089712198'-001 2.95503128589'-001	4.163
0.016	-3.10266412365'-001 -3.76632075127'-001	8.55214935895'-001 2.93222590291'-001	4.228
0.018	-3.01045165961'-001 -3.63959262402'-001	8.49264844044'-001 2.91233664299'-001	4.285
0.020	-2.92836312063'-001 -3.52720760275'-001	8.44029169606'-001 2.89471057775'-001	4.338
0.022	-2.85440670297'-001 -3.42632318469'-001	8.39360893519'-001 2.87888970088'-001	4.386
0.024	-2.78712384185'-001 -3.33486050731'-001	8.35153373595'-001 2.86454077381'-001	4.430
0.026	-2.72541481212'-001 -3.25125403488'-001	8.31326914988'-001 2.85141404438'-001	4.471
0.028	-2.66843015383'-001 -3.17429587602'-001	8.27820470942'-001 2.83931766680'-001	4.510
0.030	-2.61550024134'-001 -3.10303489543'-001	8.24586296636'-001 2.82810119668'-001	4.546
0.032	-2.56608796150'-001 -3.03670897876'-001	8.21586380716'-001 2.81764454572'-001	4.581

Type III	N = 4	$\psi_g = 1.0$	
δ	RE	$\pm IM$	$T_m \omega_g$
0.034	-2.51975599076'-001 -2.97469823308'-001	8.18789992736'-001 2.80785038412'-001	4.613
0.036	-2.47614355933'-001 -2.91649178398'-001	8.16171952873'-001 2.79863876276'-001	4.644
0.038	-2.43494961233'-001 -2.86166372367'-001	8.13711384479'-001 2.78994323711'-001	4.673
0.040	-2.39592034206'-001 -2.80985532369'-001	8.11390796167'-001 2.78170800419'-001	4.701
0.042	-2.35883981470'-001 -2.76076166490'-001	8.09195394711'-001 2.77388576003'-001	4.728
0.044	-2.32352280378'-001 -2.71412142567'-001	8.07112563340'-001 2.76643606660'-001	4.754
0.046	-2.28797426345'-001 -2.66756956073'-001	8.04486251123'-001 2.75711285522'-001	4.783
0.048	-2.25580459530'-001 -2.62528542415'-001	8.02618161688'-001 2.75037949636'-001	4.807
0.050	-2.22497072020'-001 -2.58485341270'-001	8.00832663848'-001 2.74392185977'-001	4.830
0.052	-2.19536610661'-001 -2.54612361998'-001	7.99122778721'-001 2.73771741537'-001	4.852
0.054	-2.16689646513'-001 -2.50896348112'-001	7.97482356087'-001 2.73174626389'-001	4.874
0.056	-2.13947794815'-001 -2.47325520846'-001	7.95905949955'-001 2.72599074655'-001	4.894
0.058	-2.11303566179'-001 -2.43889368328'-001	7.94388715822'-001 2.72043512108'-001	4.915
0.060	-2.08750244139'-001 -2.40578470926'-001	7.92926326587'-001 2.71506529680'-001	4.934
0.062	-2.06281782143'-001 -2.37384355583'-001	7.91514902154'-001 2.70986861175'-001	4.954
0.064	-2.03892717524'-001 -2.34299373591'-001	7.90150950405'-001 2.70483364620'-001	4.972

Type III	N=4	$\Psi_g = 1.0$	
δ	RE	$\pm IM$	$T_m \omega_g$
0.066	-2.01578099059'-001 -2.31316597754'-001	7.88831317852'-001 2.69995006831'-001	4.990
0.068	-1.99333424942'-001 -2.28429734394'-001	7.87553148031'-001 2.69520849892'-001	5.008
0.070	-1.97154590067'-001 -2.25633049071'-001	7.86313845325'-001 2.69060039936'-001	5.025
0.072	-1.95037840764'-001 -2.22921302280'-001	7.85111044752'-001 2.68611797364'-001	5.042
0.074	-1.92979736097'-001 -2.20289694502'-001	7.83942585662'-001 2.68175408597'-001	5.059
0.076	-1.90977113831'-001 -2.17733818361'-001	7.82806489202'-001 2.67750218838'-001	5.075
0.078	-1.89027061340'-001 -2.15249617344'-001	7.81700938729'-001 2.67335625787'-001	5.090
0.080	-1.87126890036'-001 -2.12833349684'-001	7.80624262734'-001 2.66931074271'-001	5.106
0.082	-1.85274112875'-001 -2.10481556620'-001	7.79574919965'-001 2.66536051357'-001	5.121
0.084	-1.83466424795'-001 -2.08191034890'-001	7.78551486299'-001 2.66150082261'-001	5.136
0.086	-1.81701685462'-001 -2.05958812156'-001	7.77552643565'-001 2.65772726732'-001	5.150
0.088	-1.79977903845'-001 -2.03782125453'-001	7.76577169174'-001 2.65403575661'-001	5.164
0.090	-1.78293224781'-001 -2.01658402086'-001	7.75623927333'-001 2.65042248317'-001	5.178
0.092	-1.76645916778'-001 -1.99585242626'-001	7.74691861061'-001 2.64688389787'-001	5.192
0.094	-1.75034361397'-001 -1.97560405773'-001	7.73779985218'-001 2.64341668659'-001	5.205
0.096	-1.73457043486'-001 -1.95581794786'-001	7.72887380183'-001 2.64001775038'-001	5.219

Type III	N=4	$\Psi_g = 1.0$	
δ	RE	$\pm IM$	$T_m \omega_g$
0.098	-1.71912542694'-001 -1.93647445452'-001	7.72013186139'-001 2.63668418676'-001	5.232
0.100	-1.70399525705'-001 -1.91755515065'-001	7.71156598326'-001 2.63341327438'-001	5.244

Type III	N = 5	$\Psi_g = 1.0$	
δ	RE	$\pm 1M$	$T_m \omega_g$
0.002	-3.54528128122'-001 -4.59559017720'-001 -4.82112185593'-001	9.86981542148'-001 5.08104476274'-001	4.650
0.004	-3.13712754651'-001 -4.00524419241'-001 -4.18675916415'-001	9.47857861745'-001 4.88478054460'-001	4.971
0.006	-2.93768419335'-001 -3.70097934909'-001 -3.85721151048'-001	9.34273966312'-001 4.82020571857'-001	5.128
0.008	-2.77144651518'-001 -3.46389963204'-001 -3.60343738837'-001	9.20763450788'-001 4.75089200161'-001	5.269
0.010	-2.64355026146'-001 -3.28214039761'-001 -3.40911987863'-001	9.10690796536'-001 4.69886955980'-001	5.381
0.012	-2.54060676052'-001 -3.12759050503'-001 -3.24227207137'-001	9.00795823331'-001 4.65935364962'-001	5.484
0.014	-2.45206741667'-001 -3.01154163797'-001 -3.12037077703'-001	8.96108109413'-001 4.62278554245'-001	5.556
0.016	-2.37643868099'-001 -2.90530428210'-001 -3.00722671018'-001	8.90505725467'-001 4.59324682997'-001	5.627
0.018	-2.30986378385'-001 -2.81213975417'-001 -2.90812235609'-001	8.85644579994'-001 4.56745100702'-001	5.690
0.020	-2.25039970788'-001 -2.72923390647'-001 -2.82002968132'-001	8.81356828955'-001 4.54455803654'-001	5.748
0.022	-2.19666775389'-001 -2.65458955728'-001 -2.74080000185'-001	8.77525051885'-001 4.52397993133'-001	5.800
0.024	-2.14765443565'-001 -2.58673863027'-001 -2.66885420409'-001	8.74064019488'-001 4.50528953177'-001	5.849

Type III	N = 5	$\psi_g = 1.0$	
δ	RE	$\pm 1M$	$T_m \omega_g$
0.026	-2.10259304495'-001 -2.52457073286'-001 -2.60299822199'-001	8.70909993246'-001 4.48816702177'-001	5.894
0.028	-2.06088960083'-001 -2.46722589760'-001 -2.54230755760'-001	8.68014110198'-001 4.47236690030'-001	5.936
0.030	-2.02207463601'-001 -2.41402469794'-001 -2.48605203466'-001	8.65338123898'-001 4.45769672489'-001	5.975
0.032	-1.98577068593'-001 -2.36442111913'-001 -2.43364504990'-001	8.62851560170'-001 4.44400291483'-001	6.013
0.034	-1.95166974721'-001 -2.31796985427'-001 -2.38460835972'-001	8.60529761110'-001 4.43116099865'-001	6.048
0.036	-1.91951722703'-001 -2.27430302883'-001 -2.33854702048'-001	8.58352506224'-001 4.41906872453'-001	6.081
0.038	-1.88910028261'-001 -2.23311327318'-001 -2.29513116443'-001	8.56303016550'-001 4.40764109600'-001	6.113
0.040	-1.86023916373'-001 -2.19414117979'-001 -2.25408249520'-001	8.54367222404'-001 4.39680671043'-001	6.144
0.042	-1.83278070337'-001 -2.15716584040'-001 -2.21516410667'-001	8.52533215068'-001 4.38650502008'-001	6.173
0.044	-1.80659331720'-001 -2.12199760579'-001 -2.17817268793'-001	8.50790829081'-001 4.37668424594'-001	6.201
0.046	-1.78156314105'-001 -2.08847247333'-001 -2.14293248515'-001	8.49131322262'-001 4.36729977362'-001	6.228
0.048	-1.75759097692'-001 -2.05644766982'-001 -2.10929055073'-001	8.47547122503'-001 4.35831288858'-001	6.254

Type III	N = 5	$\psi_g = 1.0$	
δ	RE	$\pm 1M$	$T_m \omega_g$
0.050	-1.73458988778'-001 -2.02579815250'-001 -2.07711297750'-001	8.46031631147'-001 4.34968978876'-001	6.279
0.052	-1.71248325409'-001 -1.99641379593'-001 -2.04628187574'-001	8.44579063894'-001 4.34140078851'-001	6.303
0.054	-1.69120320272'-001 -1.96819711819'-001 -2.01669292583'-001	8.43184323545'-001 4.33341967910'-001	6.326
0.056	-1.67068932820'-001 -1.94106142215'-001 -1.98825338215'-001	8.41842896462'-001 4.32572320722'-001	6.349
0.058	-1.65088763159'-001 -1.91492926380'-001 -1.96088042590'-001	8.40550767214'-001 4.31829064957'-001	6.371
0.060	-1.63174965066'-001 -1.88973118715'-001 -1.93449980601'-001	8.39304348814'-001 4.31110345835'-001	6.392
0.062	-1.61323172324'-001 -1.86540466395'-001 -1.90904469928'-001	8.38100424680'-001 4.30414496921'-001	6.413
0.064	-1.59529437653'-001 -1.84189320487'-001 -1.88445475622'-001	8.36936099320'-001 4.29740015421'-001	6.433
0.066	-1.57790180627'-001 -1.81914560908'-001 -1.86067529514'-001	8.35808757674'-001 4.29085541637'-001	6.452
0.068	-1.56102143927'-001 -1.79711532634'-001 -1.83765661692'-001	8.34716030702'-001 4.28449841475'-001	6.471
0.070	-1.54462354931'-001 -1.77575990969'-001 -1.81535341828'-001	8.33655764942'-001 4.27831791087'-001	6.490
0.072	-1.52868093952'-001 -1.75504055176'-001 -1.79372429244'-001	8.32625998050'-001 4.27230364534'-001	6.508

Type III	N = 5	$\Psi_g = 1.0$	
δ	RE	$\pm IM$	$T_m \omega_g$
0.074	-1.51316866450'-001 -1.73492167982'-001 -1.77273129644'-001	8.31624937102'-001 4.26644622756'-001	6.526
0.076	-1.49806378278'-001 -1.71537060640'-001 -1.75233957554'-001	8.30650939246'-001 4.26073703726'-001	6.543
0.078	-1.48012347792'-001 -1.69501653087'-001 -1.73170395690'-001	8.29713137187'-001 4.25308259866'-001	6.562
0.080	-1.46584017635'-001 -1.67655742607'-001 -1.71245005047'-001	8.28791403619'-001 4.24769543603'-001	6.578
0.082	-1.45190240877'-001 -1.65858019605'-001 -1.69370671840'-001	8.27892405629'-001 4.24243227168'-001	6.594
0.084	-1.43829362991'-001 -1.64106120097'-001 -1.67544872405'-001	8.27014971812'-001 4.23728700789'-001	6.610
0.086	-1.42499846334'-001 -1.62397848922'-001 -1.65765264123'-001	8.26158017215'-001 4.23225398456'-001	6.626
0.088	-1.41200259423'-001 -1.60731164160'-001 -1.64029668544'-001	8.25320534896'-001 4.22732794068'-001	6.641
0.090	-1.39929267509'-001 -1.59104163140'-001 -1.62336056535'-001	8.24501589283'-001 4.22250397594'-001	6.656
0.092	-1.38685623744'-001 -1.57515070234'-001 -1.60682534915'-001	8.23700308865'-001 4.21777751952'-001	6.670
0.094	-1.37468161960'-001 -1.55962225705'-001 -1.59067334716'-001	8.22915881217'-001 4.21314430143'-001	6.685
0.096	-1.36275789548'-001 -1.54444075848'-001 -1.57488800542'-001	8.22147547409'-001 4.20860032515'-001	6.699

Type III	N = 5	$\Psi_g = 1.0$	
δ	RE	$\pm IM$	$T_m \omega_g$
0.098	-1.35107481612'-001 -1.52959164363'-001 -1.55945381150'-001	8.21394597668'-001 4.20414184646'-001	6.713
0.100	-1.33962275373'-001 -1.51506124232'-001 -1.54435620840'-001	8.20656367181'-001 4.19976535250'-001	6.726

Type III	N=6	$\psi_g = 1.0$	
δ	RE	$\pm IM$	$T_m \omega_g$
0.002	-2.80805451403'-001 -4.02061934863'-001 -3.70673000114'-001	9.89377539805'-001 2.04019726218'-001 6.06581593347'-001	5.984
0.004	-2.50656832399'-001 -3.48170241887'-001 -3.24361187239'-001	9.61652358787'-001 1.99395461114'-001 5.91760156611'-001	6.320
0.006	-2.33175699836'-001 -2.97970168285'-001 -3.18019373131'-001	9.46257471307'-001 5.83083391601'-001 1.96556827243'-001	6.527
0.008	-2.20797514128'-001 -2.79491791138'-001 -2.97116341058'-001	9.35697703855'-001 5.76952471402'-001 1.94508431432'-001	6.678
0.010	-2.11198237044'-001 -2.65288499122'-001 -2.81159170791'-001	9.27708531544'-001 5.72213417268'-001 1.92904464162'-001	6.798
0.012	-2.03350548189'-001 -2.53764414800'-001 -2.68280264208'-001	9.21307031993'-001 5.68350696456'-001 1.91585200795'-001	6.899
0.014	-1.96708887750'-001 -2.44076495008'-001 -2.57500004314'-001	9.15979683257'-001 5.65089769571'-001 1.90463854116'-001	6.985
0.016	-1.90948601216'-001 -2.35725117964'-001 -2.48240914509'-001	9.11425302300'-001 5.62267275789'-001 1.89488050914'-001	7.061
0.018	-1.85860755606'-001 -2.28389849151'-001 -2.40134203126'-001	9.07452567553'-001 5.59778224317'-001 1.88623769831'-001	7.128
0.020	-1.81302943434'-001 -2.32930139093'-001 -2.21852930372'-001	9.03932603338'-001 1.87847667894'-001 5.57551189326'-001	7.190
0.022	-1.77173734166'-001 -2.26451859333'-001 -2.15959670655'-001	9.00774600737'-001 1.87143037208'-001 5.55535406602'-001	7.246
0.024	-1.73398320170'-001 -2.20569583304'-001 -2.10596255285'-001	8.97912246375'-001 1.86497508770'-001 5.53693520094'-001	7.297

Type III	N=6	$\psi_g = 1.0$	
δ	RE	$\pm IM$	$T_m \omega_g$
0.026	-1.69919945212'-001 -2.15185297537'-001 -2.05676523805'-001	8.95295689661'-001 1.85901670862'-001 5.51997251358'-001	7.345
0.028	-1.66694525608'-001 -2.10223192880'-001 -2.01133690847'-001	8.92886541245'-001 1.85348195675'-001 5.50424684247'-001	7.390
0.030	-1.63687137771'-001 -2.05623445087'-001 -1.96914950240'-001	8.90654632200'-001 1.84831266965'-001 5.48958493630'-001	7.432
0.032	-1.61141234166'-001 -2.01549080466'-001 -1.93585486435'-001	8.88517838779'-001 1.84500658215'-001 5.47914513314'-001	7.469
0.034	-1.58472363994'-001 -1.97523082272'-001 -1.89863479413'-001	8.86571183812'-001 1.84035050417'-001 5.46607566590'-001	7.506
0.036	-1.55953407524'-001 -1.86363190456'-001 -1.93741905037'-001	8.84742363760'-001 5.45373816742'-001 1.83595178183'-001	7.542.
0.038	-1.53568080561'-001 -1.90178262252'-001 -1.83060131760'-001	8.83017888922'-001 1.83178231036'-001 5.44205176295'-001	7.576
0.040	-1.51302625382'-001 -1.86809158258'-001 -1.79933646010'-001	8.81386426805'-001 1.82781835876'-001 5.43094826513'-001	7.609
0.042	-1.49145313822'-001 -1.83615076116'-001 -1.76966148036'-001	8.79838368812'-001 1.82403970670'-001 5.42036966137'-001	7.640
0.044	-1.47086065139'-001 -1.80579126460'-001 -1.74142545395'-001	8.78365499120'-001 1.82042898348'-001 5.41026617240'-001	7.669
0.046	-1.45116153024'-001 -1.77686943953'-001 -1.71449789876'-001	8.76960741371'-001 1.81697115873'-001 5.40059479012'-001	7.698
0.048	-1.43227970923'-001 -1.74925891344'-001 -1.68876524236'-001	8.75617957245'-001 1.81365313612'-001 5.39131807978'-001	7.725

Type III	N=6	$\Psi_g = 1.0$	
δ	RE	$\pm IM$	$T_m \omega_g$
0.050	-1.41414849870'-001 -1.72284946402'-001 -1.66412803257'-001	8.74331790132'-001 1.81046343947'-001 5.38240327849'-001	7.752
0.052	-1.39670909313'-001 -1.69754410603'-001 -1.64049869134'-001	8.73097538177'-001 1.80739195366'-001 5.37382153729'-001	7.777
0.054	-1.37990937743'-001 -1.67325706163'-001 -1.61779969789'-001	8.71911052422'-001 1.80442971610'-001 5.36554732356'-001	7.802
0.056	-1.36370294924'-001 -1.64991210258'-001 -1.59596211503'-001	8.70768654873'-001 1.80156875111'-001 5.35755794030'-001	7.826
0.058	-1.34804830890'-001 -1.62744118938'-001 -1.57492436381'-001	8.69667069270'-001 1.79880192776'-001 5.34983311172'-001	7.849
0.060	-1.33290819475'-001 -1.60578334235'-001 -1.55463121722'-001	8.68603366092'-001 1.79612284512'-001 5.34235465693'-001	7.872
0.062	-1.31824902146'-001 -1.58488370176'-001 -1.53503295536'-001	8.67574915152'-001 1.79352573607'-001 5.33510620706'-001	7.894
0.064	-1.30404041382'-001 -1.56469273828'-001 -1.51608466033'-001	8.66579346344'-001 1.79100538622'-001 5.32807297706'-001	7.915
0.066	-1.28974084262'-001 -1.54455008164'-001 -1.49714900199'-001	8.65269706301'-001 1.78784460677'-001 5.31912198748'-001	7.939
0.068	-1.27636813410'-001 -1.52566504596'-001 -1.47940046103'-001	8.64340584983'-001 1.78547846423'-001 5.31252293717'-001	7.959
0.070	-1.26366816781'-001 -1.50525035951'-001 -1.46035643138'-001	8.63772363024'-001 1.78380205889'-001 5.30812019366'-001	7.974
0.072	-1.25072429535'-001 -1.48961546569'-001 -1.44548485769'-001	8.62560739450'-001 1.78093171686'-001 5.29984375255'-001	7.998

Type III	N=6	$\Psi_g = 1.0$	
δ	RE	$\pm IM$	$T_m \omega_g$
0.074	-1.23841309640'-001 -1.47238604695'-001 -1.42925880449'-001	8.61706983942'-001 1.77874433604'-001 5.29374438622'-001	8.016
0.076	-1.22641834507'-001 -1.45564804636'-001 -1.41348524841'-001	8.60875529124'-001 1.77661021891'-001 5.28779370423'-001	8.034
0.078	-1.21472377738'-001 -1.43937523586'-001 -1.39814030445'-001	8.60065153727'-001 1.77452661472'-001 5.28198395237'-001	8.052
0.080	-1.20331435915'-001 -1.42354341657'-001 -1.38320192426'-001	8.59274732422'-001 1.77249098044'-001 5.27630796809'-001	8.070
0.082	-1.19217617492'-001 -1.40813021728'-001 -1.36864971242'-001	8.58503226376'-001 1.77050096198'-001 5.27075912363'-001	8.087
0.084	-1.18129631097'-001 -1.39311491366'-001 -1.35446476546'-001	8.57749674284'-001 1.76855437436'-001 5.26533127373'-001	8.103
0.086	-1.17097358354'-001 -1.33909137167'-001 -1.37672253175'-001	8.57298767871'-001 5.26167638956'-001 1.76716026152'-001	8.116
0.088	-1.16026436350'-001 -1.32712768016'-001 -1.36420241226'-001	8.56292931861'-001 5.25481611468'-001 1.76478351074'-001	8.136
0.090	-1.15009068264'-001 -1.31394400014'-001 -1.35027067961'-001	8.55588144517'-001 5.24971853949'-001 1.76295558034'-001	8.152
0.092	-1.14013198530'-001 -1.33666753940'-001 -1.30106429140'-001	8.54898106132'-001 1.76116374878'-001 5.24472135821'-001	8.167
0.094	-1.13037916213'-001 -1.32337847881'-001 -1.28847528300'-001	8.54222147551'-001 1.75940647441'-001 5.23982024635'-001	8.182
0.096	-1.12082367898'-001 -1.31038991865'-001 -1.27616454978'-001	8.53559643292'-001 1.75768231340'-001 5.23501115382'-001	8.197

Type III	N=6	$\psi_g = 1.0$	
δ	RE	$\pm IM$	$T_m \omega_g$
0.098	-1.11145752653'-001	8.52910008191'-001	8.212
	-1.29768913535'-001	1.75598991078'-001	
	-1.26412044720'-001	5.23029028176'-001	
0.100	-1.10227317844'-001	8.52272693366'-001	8.226
	-1.28526419067'-001	1.75432799257'-001	
	-1.25233204246'-001	5.22565406151'-001	

Type III	N = 7	$\Psi_g = 1.0$	
δ	RE	$\pm IM$	$T_m(\omega_g)$
0.002	-2.32918943093'-001 -3.30981884203'-001 -3.08562580933'-001 -3.35840657644'-001	9.93336500127'-001 3.48334990413'-001 6.82046461785'-001	7.355
0.004	-2.08331042013'-001 -2.89043100896'-001 -2.70737529820'-001 -2.93200303768'-001	9.69246117893'-001 3.38856997103'-001 6.65422530365'-001	7.721
0.006	-1.93756028835'-001 -2.64519428649'-001 -2.48667595242'-001 -2.68174011872'-001	9.54114571166'-001 3.33050081779'-001 6.54929923610'-001	7.959
0.008	-1.83813724460'-001 -2.47852860921'-001 -2.33672428290'-001 -2.51145525777'-001	9.45478093214'-001 3.29662817312'-001 6.48864281575'-001	8.118
0.010	-1.76083942726'-001 -2.35027910443'-001 -2.22110917297'-001 -2.38039217889'-001	9.38889639932'-001 3.27061095457'-001 6.44188071361'-001	8.245
0.012	-1.70524651690'-001 -2.25057339754'-001 -2.13203038926'-001 -2.27832222605'-001	9.35176679057'-001 3.25556487554'-001 6.41605279139'-001	8.336
0.014	-1.65063535344'-001 -2.16237855209'-001 -2.05205013685'-001 -2.18824495736'-001	9.30523487874'-001 3.23690908597'-001 6.38227697618'-001	8.428
0.016	-1.98309050451'-001 -1.60329140806'-001 -2.08647727824'-001 -2.11074200401'-001	6.35321618011'-001 9.26550469125'-001 3.22086134314'-001	8.509
0.018	-1.91407089686'-001 -1.55903378796'-001 -2.01505002591'-001 -2.03989380127'-001	6.32908616637'-001 9.23137199872'-001 3.20301103806'-001	8.586
0.020	-1.86080825362'-001 -1.52186095216'-001 -1.95635215132'-001 -1.97979247023'-001	6.30633220088'-001 9.20070417909'-001 3.19076233336'-001	8.650

Type III	N=7	$\Psi_g = 1.0$	
δ	RE	$\pm IM$	$T_m \omega_g$
0.022	-1.81271148093'-001 -1.90345181112'-001 -1.48814849359'-001 -1.92565742720'-001	6.28578437187'-001 3.17968228996'-001 9.17318510197'-001	8.710
0.024	-1.76887536469'-001 -1.45729447660'-001 -1.85532692427'-001 -1.87643470701'-001	6.26704401744'-001 9.14823516745'-001 3.16956189025'-001	8.764
0.026	-1.72757188146'-001 -1.42424727412'-001 -1.81197173020'-001 -1.83239063459'-001	6.23263502970'-001 9.10538528595'-001 3.15185493120'-001	8.828
0.028	-1.69043964870'-001 -1.39809977965'-001 -1.77120414657'-001 -1.79070810525'-001	6.21763082207'-001 9.08549566128'-001 3.14369508000'-001	8.875
0.030	-1.65591290428'-001 -1.37368592103'-001 -1.73336444983'-001 -1.75203711900'-001	6.20360368979'-001 9.06700291678'-001 3.13605994736'-001	8.918
0.032	-1.62446161354'-001 -1.69741595189'-001 -1.35469148059'-001 -1.71509741173'-001	6.20508161013'-001 3.13600732316'-001 9.06677498173'-001	8.948
0.034	-1.59413525682'-001 -1.33294077961'-001 -1.66438333097'-001 -1.68138315075'-001	6.19200009431'-001 9.04978227831'-001 3.12890697736'-001	8.987
0.036	-1.56557834009'-001 -1.31238878599'-001 -1.63332006894'-001 -1.64969023419'-001	6.17965172794'-001 9.03380758566'-001 3.12220000204'-001	9.024
0.038	-1.53859818325'-001 -1.29290669825'-001 -1.60400987679'-001 -1.61979609675'-001	6.16795516995'-001 9.01873468207'-001 3.11584323870'-001	9.060
0.040	-1.51303189043'-001 -1.57627004490'-001 -1.27438552489'-001 -1.59151278196'-001	6.15684189990'-001 3.10980027734'-001 9.00446591331'-001	9.094

Type III	N=7	$\Psi_g = 1.0$	
δ	RE	$\pm IM$	$T_m \omega_g$
0.042	-1.48874052269'-001 -1.25673222297'-001 -1.54994485696'-001 -1.56468014058'-001	6.14625366969'-001 8.99091846357'-001 3.10404012442'-001	9.126
0.044	-1.46560462166'-001 -1.52490053511'-001 -1.23986672024'-001 -1.53916062092'-001	6.13614053586'-001 3.09853617415'-001 8.97802151023'-001	9.157
0.046	-1.44352073716'-001 -1.22371960952'-001 -1.50102131105'-001 -1.51483521888'-001	6.12645936901'-001 8.96571403508'-001 3.09326542367'-001	9.187
0.048	-1.42239868812'-001 -1.20823033099'-001 -1.47820634542'-001 -1.49160029931'-001	6.11717266180'-001 8.95394310224'-001 3.08820784238'-001	9.216
0.050	-1.19334573166'-001 -1.40215939304'-001 -1.45636727175'-001 -1.46936506677'-001	8.94266250605'-001 6.10824759162'-001 3.08334588597'-001	9.244
0.052	-1.17901889276'-001 -1.38273311492'-001 -1.43542622322'-001 -1.44804953256'-001	8.93183167213'-001 6.09965527707'-001 3.07866409676'-001	9.271
0.054	-1.36405805675'-001 -1.16520819734'-001 -1.41531424215'-001 -1.42758287526'-001	6.09137016912'-001 8.92141479853'-001 3.07414878873'-001	9.296
0.056	-1.15187656258'-001 -1.34607918953'-001 -1.39596997367'-001 -1.40790209342'-001	8.91138013940'-001 6.08336955367'-001 3.06978778448'-001	9.321
0.058	-1.13899078951'-001 -1.32874731396'-001 -1.37733858937'-001 -1.38895090022'-001	8.90169940329'-001 6.07563316211'-001 3.06557020187'-001	9.346
0.060	-1.31201825137'-001 -1.12652104934'-001 -1.35937089519'-001 -1.37067880281'-001	6.06814280753'-001 8.89234729231'-001 3.06148627402'-001	9.369

Type III	N = 7	$\Psi_g = 1.0$	
δ	RE	$\pm 1M$	$T_m \omega_g$
0.062	-1.29585219573'-001 -1.11444043556'-001 -1.34202259142'-001 -1.35304034011'-001	6.06088213408'-001 8.88330108443'-001 3.05752720597'-001	9.392
0.064	-1.28021314515'-001 -1.10272459802'-001 -1.32525364243'-001 -1.33599443470'-001	6.05383636266'-001 8.87454030220'-001 3.05368504400'-001	9.414
0.066	-1.26506844594'-001 -1.30902774952'-001 -1.09135142204'-001 -1.31950385170'-001	6.04699210373'-001 3.04995257731'-001 8.86604643300'-001	9.436
0.068	-1.25038837628'-001 -1.08030076754'-001 -1.29331190321'-001 -1.30353473294'-001	6.04033718220'-001 8.85780267490'-001 3.04632324733'-001	9.457
0.070	-1.23614581466'-001 -1.27807599325'-001 -1.06955423780'-001 -1.28805620378'-001	6.03386049734'-001 3.04279106422'-001 8.84979374896'-001	9.477
0.072	-1.22231594185'-001 -1.05909498187'-001 -1.26329248691'-001 -1.27304003319'-001	6.02755190259'-001 8.84200570727'-001 3.03935054959'-001	9.497
0.074	-1.20887599227'-001 -1.04890752969'-001 -1.24893613753'-001 -1.25846034247'-001	6.02140209554'-001 8.83442579565'-001 3.03599668194'-001	9.517
0.076	-1.19580502844'-001 -1.03897764006'-001 -1.23498374363'-001 -1.24429335068'-001	6.01540252723'-001 8.82704231513'-001 3.03272483790'-001	9.536
0.078	-1.18308375179'-001 -1.02929217992'-001 -1.22141393265'-001 -1.23051715535'-001	6.00954532256'-001 8.81984451389'-001 3.02953076083'-001	9.554
0.080	-1.17069433993'-001 -1.01983900400'-001 -1.20820697404'-001 -1.21711154054'-001	6.00382321580'-001 8.81282248491'-001 3.02641051698'-001	9.572

Type III	N=7	$\psi_g = 1.0$	
δ	RE	$\pm IM$	$T_m \omega_g$
0.082	-1.15862029354'-001 -1.01060686153'-001 -1.19534461643'-001 -1.20405780806'-001	5.99822948007'-001 2.80596707990'-001 3.02336046450'-001	9.590
0.084	-1.14684631557'-001 -1.00158531433'-001 -1.18280994320'-001 -1.19133863092'-001	5.99275788558'-001 3.79926984046'-001 3.02037722708'-001	9.608
0.086	-1.13535819510'-001 -9.92764652910'-002 -1.17058724751'-001 -1.17893792205'-001	5.98740264642'-001 8.79272292077'-001 3.01745766858'-001	9.625
0.088	-1.12414270167'-001 -9.84135834442'-002 -1.15866191816'-001 -1.16684071945'-001	5.98215837603'-001 8.78631904001'-001 3.01459886661'-001	9.641
0.090	-1.11318750638'-001 -9.75690422129'-002 -1.14702034087'-001 -1.15503308443'-001	5.97702006158'-001 8.78005142713'-001 3.01179810005'-001	9.658
0.092	-9.67420531561'-002 -1.13564980980'-001 -1.10248109369'-001 -1.14350201002'-001	3.77391377071'-001 3.00905282544'-001 5.97198301741'-001	9.674
0.094	-9.59318781825'-002 -1.12453845020'-001 -1.09201269814'-001 -1.13223534107'-001	8.76790017745'-001 3.00636066411'-001 5.96704286589'-001	9.690
0.096	-9.51378258942'-002 -1.11367514561'-001 -1.08177223585'-001 -1.12122170081'-001	3.76200515144'-001 3.00371938844'-001 5.96219550927'-001	9.705
0.098	-9.41945792256'-002 -1.10327023205'-001 -1.07156759019'-001 -1.11072579733'-001	8.74856088278'-001 2.99796395339'-001 5.95082764281'-001	9.725
0.100	-9.34334037811'-002 -1.09286728537'-001 -1.06176043249'-001 -1.10018021343'-001	8.74298721043'-001 2.99545992380'-001 5.94624093515'-001	9.739

Type III	N= 8	$\Psi_g = 1.0$	
δ	RE	$\pm IM$	$T_m \omega_g$
0.002	-1.98255004201'-001 -2.84584805682'-001 -2.96134166725'-001 -2.60732173916'-001	9.97122519726'-001 4.43213399827'-001 1.48122571078'-001 7.31510440848'-001	8.743
0.004	-1.75722445489'-001 -2.47285934202'-001 -2.28133382898'-001 -2.55680992545'-001	9.73313627168'-001 4.32843338058'-001 7.13246288535'-001 1.44904338606'-001	9.169
0.006	-1.64511808842'-001 -2.27222770045'-001 -2.10969658569'-001 -2.34086871278'-001	9.62274128531'-001 4.28278652198'-001 7.05439133133'-001 1.43432660851'-001	9.398
0.008	-1.56459506899'-001 -2.13121363020'-001 -1.98766853920'-001 -2.19044915059'-001	9.54614405342'-001 4.24919548990'-001 6.99870513461'-001 1.42323956527'-001	9.566
0.010	-1.50417621280'-001 -2.03511706321'-001 -1.90011623417'-001 -2.08075515424'-001	9.48733971512'-001 4.22607227287'-001 6.95149176405'-001 1.41686954172'-001	9.693
0.012	-1.45176281680'-001 -1.94503330833'-001 -1.82143874379'-001 -1.98653977361'-001	9.44020552797'-001 4.20357636594'-001 6.91640444689'-001 1.40905731863'-001	9.805
0.014	-1.40722987457'-001 -1.86949221494'-001 -1.75509822511'-001 -1.90769944107'-001	9.40078920758'-001 4.18436919546'-001 6.88661883221'-001 1.40238275199'-001	9.900
0.016	-1.36847256730'-001 -1.80450130392'-001 -1.69775685397'-001 -1.83998834156'-001	9.36694888682'-001 4.16759250067'-001 6.86071768869'-001 1.39655015395'-001	9.984
0.018	-1.33413338172'-001 -1.74751044648'-001 -1.64727078372'-001 -1.78070081394'-001	9.33731919147'-001 4.15268598459'-001 6.83778507300'-001 1.39136614387'-001	10.059

Type III	N = 8	$\psi_g = 1.0$	
δ	RE	$\pm 1M$	$T_m \omega_g$
0.020	-1.30328511710'-001 -1.69679208226'-001 -1.60218097473'-001 -1.72800745469'-001	9.31097748449'-001 4.13926358069'-001 6.81719491287'-001 1.38669728829'-001	10.127
0.022	-1.56144944271'-001 -1.27526606098'-001 -1.65112207724'-001 -1.68061406668'-001	6.79850039072'-001 9.28727182891'-001 4.12704794318'-001 1.38244762198'-001	10.189
0.024	-1.24958727443'-001 -1.60960162419'-001 -1.52431140770'-001 -1.63757175040'-001	9.26572490276'-001 4.11583307465'-001 6.78137148236'-001 1.37854580098'-001	10.246
0.026	-1.49018713888'-001 -1.22587713988'-001 -1.57155148962'-001 -1.59816431144'-001	6.76555752270'-001 9.24597702571'-001 4.10546176663'-001 1.37493733520'-001	10.299
0.028	-1.20384653623'-001 -1.53644596980'-001 -1.45862669466'-001 -1.56183806474'-001	9.22775060688'-001 4.09581130039'-001 6.75086367555'-001 1.37157965797'-001	10.348
0.030	-1.18326605279'-001 -1.50386994224'-001 -1.42927386463'-001 -1.52815623122'-001	9.21082704041'-001 4.09678402640'-001 6.73713547207'-001 1.36843888093'-001	10.394
0.032	-1.16395057034'-001 -1.47348992370'-001 -1.40184183627'-001 -1.49676825906'-001	9.19503118519'-001 4.07830096785'-001 6.72424334580'-001 1.36548757932'-001	10.437
0.034	-1.14574853247'-001 -1.44503402666'-001 -1.37609625450'-001 -1.46738858573'-001	9.18022061363'-001 4.07029733324'-001 6.71210061129'-001 1.36270324503'-001	10.478
0.036	-1.12853425799'-001 -1.41827768730'-001 -1.35184317831'-001 -1.43978154232'-001	9.16627798652'-001 4.06271930624'-001 6.70060775785'-001 1.36006717366'-001	10.517
0.038	-1.11220236777'-001 -1.39303331875'-001 -1.32892028378'-001 -1.41375040446'-001	9.15310550601'-001 4.05552169490'-001 6.68969917020'-001 1.35756365311'-001	10.554

Type III	N = 8	$\Psi_g = 1.0$	
δ	RE	$\pm IM$	$T_m \omega_g$
0.040	-1.09666363180'-001 -1.36914262333'-001 -1.30719035980'-001 -1.38912928280'-001	9.14062084790'-001 4.04866618275'-001 6.67931508157'-001 1.35517935767'-001	10.589
0.042	-1.08184180313'-001 -1.34647080777'-001 -1.28653636661'-001 -1.36577701102'-001	9.12875406790'-001 4.04211999197'-001 6.66940446725'-001 1.35290288334'-001	10.623
0.044	-1.06767121488'-001 -1.32490216809'-001 -1.26685768584'-001 -1.34357248422'-001	9.11744523790'-001 4.03585486736'-001 6.65992342052'-001 1.35072439676'-001	10.655
0.046	-1.05409491833'-001 -1.30433664912'-001 -1.24806716929'-001 -1.32241103958'-001	9.10664267020'-001 4.02984628119'-001 6.65083391417'-001 1.34863536056'-001	10.686
0.048	-1.04106317576'-001 -1.28468714585'-001 -1.23008884868'-001 -1.30220161405'-001	9.09630142876'-001 4.02407279133'-001 6.64210275332'-001 1.34662830893'-001	10.716
0.050	-1.02853231369'-001 -1.26587737981'-001 -1.21285610137'-001 -1.28286449904'-001	9.08638225461'-001 4.01851556206'-001 6.63370082613'-001 1.34469668239'-001	10.745
0.052	-1.01646375177'-001 -1.24784016796'-001 -1.19631017180'-001 -1.26432952809'-001	9.07685062804'-001 4.01315794068'-001 6.62560243021'-001 1.34283468018'-001	10.773
0.054	-1.00482325473'-001 -1.23051604140'-001 -1.18039898720'-001 -1.24653462088'-001	9.06767604785'-001 4.00798514122'-001 6.61778478869'-001 1.34103715129'-001	10.799
0.056	-9.93580290560'-002 -1.21385210943'-001 -1.16507618055'-001 -1.22942458911'-001	9.05883143572'-001 4.00298397217'-001 6.61022759479'-001 1.33929949923'-001	10.825
0.058	-9.82705093265'-002 -1.19780524459'-001 -1.15030024979'-001 -1.21295988734'-001	9.05029007870'-001 3.99813532297'-001 6.60290706874'-001 1.33761101519'-001	10.850

Type III	N=8	$\Psi_g = 1.0$	
δ	RE	$\pm 1M$	$T_m \omega_g$
0.060	-9.73048485212'-002 -1.18284765627'-001 -1.13849106314'-001 -1.19739294318'-001	9.04167656365'-001 3.99460661665'-001 6.59765459022'-001 1.33632992682'-001	10.873
0.062	-9.62811010941'-002 -1.16787725590'-001 -1.12463477744'-001 -1.18204704955'-001	9.03368836749'-001 3.99001752026'-001 6.59073606192'-001 1.33473752248'-001	10.897
0.064	-9.52879032775'-002 -1.15340636654'-001 -1.11122810034'-001 -1.16721795243'-001	9.02594761759'-001 3.98556153898'-001 6.58401787143'-001 1.33319149587'-001	10.920
0.066	-9.43234104707'-002 -1.13940359417'-001 -1.09824315561'-001 -1.15287301676'-001	9.01843841199'-001 3.98123057094'-001 6.57748771977'-001 1.33168902758'-001	10.942
0.068	-9.33859419311'-002 -1.12584040628'-001 -1.08565458810'-001 -1.13898258671'-001	9.01114630746'-001 3.97701723970'-001 6.57113441095'-001 1.33022755046'-001	10.964
0.070	-9.24739627517'-002 -1.11269078534'-001 -1.07343925560'-001 -1.12551963117'-001	9.00405816363'-001 3.97291480607'-001 6.56494772047'-001 1.32880471836'-001	10.985
0.072	-9.15860663445'-002 -1.09993094648'-001 -1.06157598602'-001 -1.11245944543'-001	8.99716197229'-001 3.96891709446'-001 6.55891829319'-001 1.32741838192'-001	11.006
0.074	-9.07209613179'-002 -1.08753909182'-001 -1.05004536155'-001 -1.09977939301'-001	8.99044674872'-001 3.96501843934'-001 6.55303754862'-001 1.32606656815'-001	11.026
0.076	-8.98774590969'-002 -1.07549518929'-001 -1.03882952695'-001 -1.08745868114'-001	8.98390241233'-001 3.96121362330'-001 6.54729759001'-001 1.32474746053'-001	11.046
0.078	-8.90544624890'-002 -1.06378079516'-001 -1.02791202495'-001 -1.07547816690'-001	8.97751968932'-001 3.95749783234'-001 6.54169113281'-001 1.32345938255'-001	11.065

Type III	N = 8	$\Psi_g = 1.0$	
δ	RE	$\pm IM$	$Im \omega_g$
0.080	-8.82281491467'-002 -1.05210689944'-001 -1.01701474647'-001 -1.06354524440'-001	8.96897141141'-001 3.95284474613'-001 6.53452218088'-001 1.32185906379'-001	11.087
0.082	-8.74437326056'-002 -1.04100857065'-001 -1.00665596841'-001 -1.05220041547'-001	8.96292267447'-001 3.94930996390'-001 6.52918933498'-001 1.32063386474'-001	11.106
0.084	-8.66769707282'-002 -1.03019206530'-001 -9.96552911638'-002 -1.04114632890'-001	8.95701022200'-001 3.94585110538'-001 6.52397010621'-001 1.31943514141'-001	11.124
0.086	-8.59270545334'-002 -1.01964386467'-001 -9.86693481973'-002 -1.03036896748'-001	8.95122739206'-001 3.94246466385'-001 6.51885920510'-001 1.31826167137'-001	11.141
0.088	-8.51932297082'-002 -1.00935139311'-001 -9.77066422129'-002 -1.01985529633'-001	8.94556797557'-001 3.93914736545'-001 6.51385169993'-001 1.31711231359'-001	11.158
0.090	-8.44747927004'-002 -9.99302936107'-002 -9.67661239956'-002 -1.00959317591'-001	8.94002620818'-001 3.93589616015'-001 6.50894299724'-001 1.31598600474'-001	11.175
0.092	-8.37710855303'-002 -9.89487561456'-002 -9.58468138306'-002 -9.99571279899'-002	8.93459670573'-001 3.93270819302'-001 6.50412879797'-001 1.31488174953'-001	11.192
0.094	-8.30814924557'-002 -9.79895054334'-002 -9.49477956165'-002 -9.89779026568'-002	8.92927444376'-001 3.92958079370'-001 6.49940508476'-001 1.31379861708'-001	11.208
0.096	-8.24054363293'-002 -9.70515850240'-002 -9.40682113469'-002 -9.80206513159'-002	8.92405472157'-001 3.92651145674'-001 6.49476809063'-001 1.31273573381'-001	11.224
0.098	-8.17423750486'-002 -9.61340988069'-002 -9.32072563919'-002 -9.70844460971'-002	8.91893313983'-001 3.92349783171'-001 6.49021428013'-001 1.31169228022'-001	11.240

Type III	N = 9	$\psi_g = 1.0$	
δ	RE	$\pm IM$	$T_m \omega_g$
0.002	-1.71502625344'-001 -2.29075078503'-001 -2.55257072386'-001 -2.49718667939'-001 -2.56158700257'-001	9.99515281997'-001 7.66478498059'-001 2.64446979305'-001 5.21151612222'-001	10.180
0.004	-1.53706976103'-001 -2.00531080409'-001 -2.21109036127'-001 -2.15487424342'-001 -2.22743096034'-001	9.77486075197'-001 7.53136401152'-001 2.57320877208'-001 5.10799474112'-001	10.607
0.006	-1.43904200439'-001 -1.85356659727'-001 -2.03293912909'-001 -1.98373234249'-001 -2.04710685677'-001	9.67551641130'-001 7.45141766551'-001 2.54151722751'-001 5.04757383606'-001	10.850
0.008	-1.36744799274'-001 -1.74429474664'-001 -1.90489607641'-001 -1.86079109364'-001 -1.91751560647'-001	9.59568494858'-001 7.38699509886'-001 2.51656026136'-001 4.99975741957'-001	11.039
0.010	-1.31313137157'-001 -1.66174981270'-001 -1.80826915312'-001 -1.76802936068'-001 -1.81972999417'-001	9.54543767570'-001 7.34561668534'-001 2.50017534338'-001 4.96857105514'-001	11.177
0.012	-1.26842873851'-001 -1.59435724665'-001 -1.72962964789'-001 -1.69249180766'-001 -1.74016916666'-001	9.50473862427'-001 7.31177657552'-001 2.48679013435'-001 4.94309870061'-001	11.292
0.014	-1.23038319165'-001 -1.53741377217'-001 -1.66339896394'-001 -1.62882872814'-001 -1.67318127620'-001	9.47056826000'-001 7.28311211744'-001 2.47545788101'-001 4.92153695057'-001	11.390
0.016	-1.20140249415'-001 -1.49028494750'-001 -1.60795221451'-001 -1.57563739162'-001 -1.61708086504'-001	9.45054651205'-001 7.26617582695'-001 2.46820736684'-001 4.90803891793'-001	11.465

Type III	N=9	$\psi_g = 1.0$	
δ	RE	$\pm 1M$	$T_m \omega_g$
0.018	-1.17163523973'-001 -1.44655580254'-001 -1.55749123620'-001 -1.52704457330'-001 -1.56607389283'-001	9.42377357445'-001 7.24334394559'-001 2.45923464980'-001 4.89094290766'-001	11.543
0.020	-1.14489590469'-001 -1.40748286533'-001 -1.51252668249'-001 -1.48371475545'-001 -1.52063356312'-001	9.39999471641'-001 7.22293113078'-001 2.45120962431'-001 4.87565584743'-001	11.614
0.022	-1.37217127256'-001 -1.12060831831'-001 -1.47199720141'-001 -1.44463248085'-001 -1.47968440579'-001	7.20446059881'-001 9.37860989397'-001 2.44394607184'-001 4.86182151421'-001	11.678
0.024	-1.09834735933'-001 -1.33996166580'-001 -1.43512070852'-001 -1.40904963628'-001 -1.44243395676'-001	9.35918152274'-001 7.18758392104'-001 2.43730789749'-001 4.84917969443'-001	11.737
0.026	-1.07779030514'-001 -1.31035416038'-001 -1.40130444190'-001 -1.37639925695'-001 -1.40828161366'-001	9.34138092864'-001 7.17203877110'-001 2.43119261504'-001 4.83753451030'-001	11.792
0.028	-1.05868630686'-001 -1.28296074690'-001 -1.37008849659'-001 -1.34624144353'-001 -1.37676142113'-001	9.32495508609'-001 7.15762273350'-001 2.42552110030'-001 4.82673481572'-001	11.843
0.030	-1.04083647413'-001 -1.25747421324'-001 -1.34110895142'-001 -1.31822809592'-001 -1.34750469777'-001	9.30970516805'-001 7.14417633339'-001 2.42023099687'-001 4.81666152566'-001	11.891

Type III	N=9	$\Psi_g = 1.0$	
δ	RE	$\pm 1M$	$T_m \omega_g$
0.032	-1.02408043895'-001 -1.29207904854'-001 -1.23364718485'-001 -1.31407294874'-001 -1.32021477850'-001	9.29547218770'-001 4.80721917018'-001 7.13157174316'-001 2.41527228733'-001	11.936
0.034	-1.00828695646'-001 -1.26756548198'-001 -1.21127748284'-001 -1.28874134673'-001 -1.29464943253'-001	9.28212702504'-001 4.79832998033'-001 7.11970487260'-001 2.41060421598'-001	11.978
0.036	-9.93347261538'-002 -1.19019778693'-001 -1.26491637533'-001 -1.24449814550'-001 -1.27060835486'-001	9.26956345935'-001 7.10848987837'-001 2.40619315533'-001 4.78992981021'-001	12.019
0.038	-9.79170194369'-002 -1.17026803411'-001 -1.24243262857'-001 -1.22271871904'-001 -1.24792403352'-001	9.25769306290'-001 7.09785511033'-001 2.40201101711'-001 4.78196510579'-001	12.057
0.040	-9.65678558750'-002 -1.15136980036'-001 -1.22115038066'-001 -1.20209343513'-001 -1.22645497817'-001	9.24644145849'-001 7.08774014158'-001 2.39803409830'-001 4.77439068112'-001	12.093
0.042	-1.18250827960'-001 -9.52806455316'-002 -1.13340209529'-001 -1.20095055787'-001 -1.20608061920'-001	4.76716809655'-001 9.23574551973'-001 7.07809357897'-001 2.39424222866'-001	12.128
0.044	-9.39936129034'-002 -1.11343865334'-001 -1.18128045182'-001 -1.16268739141'-001 -1.18633967242'-001	9.22582024450'-001 7.06958535011'-001 2.38987981287'-001 4.75848772207'-001	12.163
0.046	-1.09718658303'-001 -9.28170425905'-002 -1.16297824192'-001 -1.14496233306'-001 -1.16787880736'-001	7.06074844155'-001 9.21607628123'-001 2.38644034918'-001 4.75194206822'-001	12.195

Type III	N=9	$\Psi_g = 1.0$	
δ	RE	$\pm IM$	$T_m \omega_g$
0.048	-9.16872609123'-002 -1.12801548334'-001 -1.08162928897'-001 -1.14548808935'-001 -1.15023940100'-001	9.20674694229'-001 4.74565658886'-001 7.05226403079'-001 2.38313792899'-001	12.226
0.050	-9.06005276956'-002 -1.06671015987'-001 -1.12874271814'-001 -1.11178270272'-001 -1.13335329952'-001	9.19779688840'-001 7.04410312785'-001 2.37996142292'-001 4.73961010343'-001	12.256
0.052	-8.95535395430'-002 -1.05237924963'-001 -1.11268283519'-001 -1.09620744210'-001 -1.11716044408'-001	9.18919506952'-001 7.03624019807'-001 2.37690101611'-001 4.73378394032'-001	12.284
0.054	-8.85433623480'-002 -1.03859225288'-001 -1.09725591890'-001 -1.08123957998'-001 -1.10160763856'-001	9.18091405645'-001 7.02865261752'-001 2.37394800166'-001 4.72816153779'-001	12.312
0.056	-8.75673802948'-002 -1.02530969423'-001 -1.08241522612'-001 -1.06683450198'-001 -1.08664754760'-001	9.17292949631'-001 7.02132026927'-001 2.37109462480'-001 4.72272815299'-001	12.339
0.058	-8.66232490444'-002 -1.01249624509'-001 -1.06811897753'-001 -1.05295230751'-001 -1.07223786621'-001	9.16521968524'-001 7.01422515849'-001 2.36833394418'-001 4.71747059011'-001	12.365
0.060	-8.57088612574'-002 -1.00012017763'-001 -1.05432968398'-001 -1.03955718398'-001 -1.05834063765'-001	9.15776519007'-001 7.00735115664'-001 2.36565972860'-001 4.71237700636'-001	12.390
0.062	-8.48223144312'-002 -9.88152866121'-002 -1.04101357458'-001 -1.02661686236'-001 -1.04492167440'-001	9.15054853889'-001 7.00068371654'-001 2.36306635328'-001 4.70743671925'-001	12.414

Type III	N = 9	$\psi_g = 1.0$	
δ	RE	$\pm IM$	$T_m \omega_g$
0.064	-8.39618856480'-002 -9.76568417027'-002 -1.02814012957'-001 -1.01410216532'-001 -1.03195008152'-001	9.14355398967'-001 6.99420970559'-001 2.36054873411'-001 4.70264006726'-001	12.438
0.066	-8.31260085433'-002 -9.65343323405'-002 -1.01568167018'-001 -1.00198662163'-001 -1.01939784316'-001	9.13676727891'-001 6.98791719478'-001 2.35810225131'-001 4.69797827449'-001	12.461
0.068	-8.23132563710'-002 -9.54456178151'-002 -1.00361301938'-001 -9.90246149304'-002 -1.00723948079'-001	9.13017547056'-001 6.98179534578'-001 2.35572269783'-001 4.69344336103'-001	12.483
0.070	-8.15223241152'-002 -9.43887442627'-002 -9.91911214878'-002 -9.78858768685'-002 -9.95451754898'-002	9.12376677716'-001 6.97583426168'-001 2.35340623345'-001 4.68902802732'-001	12.505
0.072	-8.07520168892'-002 -9.33619216667'-002 -9.80555250571'-002 -9.67804378561'-002 -9.84013411726'-002	9.11753044453'-001 6.97002488538'-001 2.35114933988'-001 4.68472560460'-001	12.526
0.074	-9.23635073195'-002 -8.00012360705'-002 -9.69525867004'-002 -9.57064535614'-002 -9.72904963492'-002	6.96435891412'-001 9.11145662656'-001 2.34894878944'-001 4.68052995828'-001	12.547
0.076	-9.13919903906'-002 -7.92689701011'-002 -9.58805358932'-002 -9.46622278862'-002 -9.62108498643'-002	6.95882871347'-001 9.10553628099'-001 2.34680160833'-001 4.67643545038'-001	12.567
0.078	-9.04459771300'-002 -7.85542848462'-002 -9.48377416280'-002 -9.36461982201'-002 -9.51607517163'-002	6.95342723371'-001 9.09976109855'-001 2.34470505515'-001 4.67243687905'-001	12.587

Type III	N = 9	$\psi_g = 1.0$	
δ	RE	$\pm IM$	$T_m \omega_g$
0.080	-8.95241793805'-002 -7.78563175627'-002 -9.38226980070'-002 -9.26569212934'-002 -9.41386787054'-002	6.94814798811'-001 9.09412341513'-001 2.34265659678'-001 4.66852943784'-001	12.607
0.082	-8.86254048564'-002 -7.71742672150'-002 -9.28340119499'-002 -9.16930617459'-002 -9.31432217403'-002	6.94298494622'-001 9.08861615615'-001 2.34065388577'-001 4.66470867628'-001	12.626
0.084	-8.77485468955'-002 -7.65073905532'-002 -9.18703919670'-002 -9.07533810459'-002 -9.21730747214'-002	6.93793252878'-001 9.08323276359'-001 2.33869474319'-001 4.66097046068'-001	12.644
0.086	-8.68925772302'-002 -7.58549960337'-002 -9.09306389258'-002 -8.98367289919'-002 -9.12270248919'-002	6.93298555056'-001 9.07796716580'-001 2.33677714470'-001 4.65731095170'-001	12.662
0.088	-8.60565384646'-002 -7.52164389196'-002 -9.00136370801'-002 -8.89420351159'-002 -9.03039440564'-002	6.92813917955'-001 9.07281371532'-001 2.33489920309'-001 4.65372657316'-001	12.680
0.090	-8.52395376969'-002 -7.45911176564'-002 -8.91183469030'-002 -8.80683015676'-002 -8.94027810609'-002	6.92338892106'-001 9.06776715536'-001 2.33305915874'-001 4.65021398617'-001	12.697
0.092	-8.44407400618'-002 -7.39784699217'-002 -8.82437979708'-002 -8.72145970934'-002 -8.85225548118'-002	6.91873056421'-001 9.06282259384'-001 2.33125536579'-001 4.64677007578'-001	12.715

Type III	N = 9	$\psi_g = 1.0$	
δ	RE	$\pm IM$	$T_m \omega_g$
0.094	-8.36593651256'-002 -7.33779689617'-002 -8.73890833391'-002 -8.63800507505'-002 -8.76623483164'-002	6.91416018104'-001 9.05797545514'-001 2.32948628713'-001 4.64339191996'-001	12.731
0.096	-8.28946826478'-002 -7.27891207498'-002 -8.65533539327'-002 -8.55638468784'-002 -8.68213032372'-002	6.90967408751'-001 9.05322146369'-001 2.32775047913'-001 4.64007679024'-001	12.748
0.098	-8.21460056613'-002 -7.22114614655'-002 -8.57358136964'-002 -8.47652211669'-002 -8.59986149044'-002	6.90526882372'-001 9.04855661822'-001 2.32604658821'-001 4.63682211892'-001	12.764
0.100	-8.14126902224'-002 -7.16445549806'-002 -8.49357154802'-002 -8.39834552171'-002 -8.51935280552'-002	6.90094115296'-001 9.04397715774'-001 2.32437334169'-001 4.63362549053'-001	12.780

APPENDIX VI

Solution of the Equiripple Approximation of the Bandpass-phase function $B_b(\omega)$ "TYPE III-general"

1. Relations

$$B_b(\omega) = B(\omega) - B_0$$

$$B_0 = -k \cdot \pi \quad k = \text{constant},$$

We define $\phi_b(\omega) = B_b(\omega) - T_m \cdot \omega$.

In the approximation interval $\phi_b(\omega)$ has N peaks at which

$$\phi_b(\omega) = \pm \delta$$

2. Parameters

Upper band edge frequency

$$\omega_h$$

lower band edge frequency

$$\omega_l$$

center frequency

$$\omega_0 = \frac{\omega_h + \omega_l}{2}$$

Relative bandwidth

$$\gamma = \frac{\omega_h - \omega_l}{\omega_0}$$

Normalized frequency

$$\psi = \omega / \omega_0$$

Delay-center frequency product

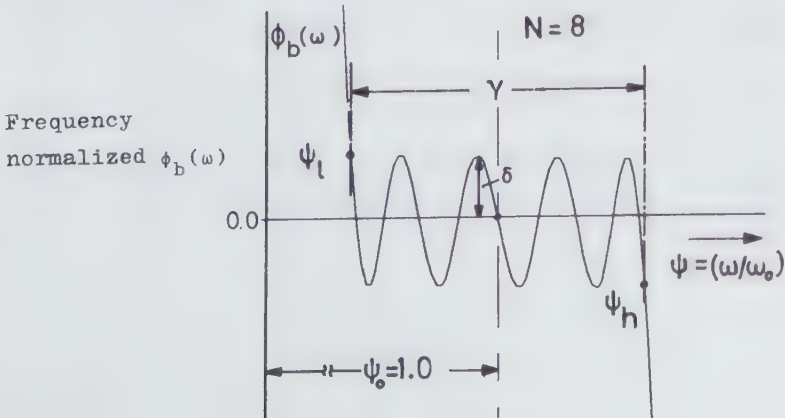
$$T_m \omega_0$$

Normalized poles

$$RE \pm j IM = (P_i / \omega_0)$$

Normalized ripple magnitude

$$\delta$$



γ K	N = 2 RE	$\psi_0 = 1.0$	Type III-general $\pm IM$	$T_{II} \psi_0$
$B_0 = -K \cdot \pi$		$\delta = 0.01$		
0.898 0.057	-8.29247610673'-001		1.05782845814'+000	1.298
0.700 0.113	-6.57543735644'-001		1.02589620970'+000	1.573
0.499 0.246	-4.73568089634'-001		1.00771097606'+000	2.096
0.300 0.617	-2.85354436233'-001		1.00108598889'+000	3.364

γ K	N = 2 RE	$\psi_0 = 1.0$	Type III-general $\pm IM$	$T_{II} \omega_0$
$B_0 = -K \cdot \pi$		$\delta = 0.03$		
0.899 0.144	-5.47765912037'-001		1.01906988077'+000	1.725
0.699 0.256	-4.28410691777'-001		1.00747914991'+000	2.148
0.500 0.483	-3.07119877005'-001		1.00204175210'+000	2.930
0.300 1.057	-1.84482568246'-001		1.00027127052'+000	4.798

γ K	N=2	$\Psi_0=1.0$	Type III-general	
	RE		$\pm IM$	$T_m \omega_0$
	$B_0 = -K \cdot \pi$		$\delta = 0.05$	
0.900 0.210	-4.38463640039'	-001	1.01083768980'+000	1.993
0.699 0.355	-3.41935150867'	-001	1.00409730856'+000	2.506
0.500 0.637	-2.44715190755'	-001	1.00109049853'+000	3.446
0.300	-1.46955394157'	-001	1.00014301325'+000	5.675

γ K	N = 2	$\psi_0 = 1.0$	Type III-general	
	RE		$\pm IM$	$T_m \omega_0$
	$B_0 = -K \cdot \pi$		$\delta = 0.07$	
0.900 0.335	-3.08052359109'	-001	1.00481610728'+000	2.458
0.699 0.533	-2.39818233564'	-001	1.00176695216'+000	3.120
0.500 0.904	-1.71485455136'	-001	1.00046107953'+000	4.323
0.300 1.794	-1.02961554439'	-001	1.00000001117'+000	7.155

γ K	N=2 RE	$\psi_0=1.0$	Type III-general $\pm IM$	$T_m \omega_0$
$B_0 = -K \cdot \pi$		$\delta = 0.1$		
0.900 0.265	-3.72737680063'-001		1.00735718041'+000	2.203
0.700 0.434	-2.90519614591'-001		1.00274038734'+000	2.782
0.500 0.757	-2.07736883898'-001		1.00072028372'+000	3.842
0.300 1.540	-1.24712266548'-001		1.00009381626'+000	6.346

γ K	N=4 RE	$\psi_0=1.0$	Type III-general $\pm IM$	$T_m \omega_0$
$B_0 = -K \cdot \pi$		$\delta = 0.01$		
0.892 0.260	-3.77321518670'-001 -3.63022303158'-001		7.04298074215'-001 1.31115964876'+000	3.541
0.698 0.510	-2.90427635446'-001 -2.85570336037'-001		7.66050443453'-001 1.24184299586'+000	4.430
0.498 1.010	-2.06127833029'-001 -2.04894425766'-001		8.31371263485'-001 1.17158220377'+000	6.098
0.300 2.230	-1.23750808503'-001 -1.23591849384'-001		8.98136156837'-001 1.10298417771'+000	10.019

γ K	N = 4 RE	$\psi_0 = 1.0$	Type III-general $\pm IM$	$T_m \omega_0$
$B_0 = -K \cdot \pi$		$\delta = 0.03$		
0.894 0.460	-2.64775395895'-001 -2.59551033985'-001		7.39242523323'-001 1.26681417431'+000	4.300
0.697 0.800	-2.05284203967'-001 -2.03237176971'-001		7.94825169844'-001 1.20659495198'+000	5.442
0.499 1.440	-1.46696802734'-001 -1.46017626641'-001		8.52283038625'-001 1.14752873167'+000	7.518
0.300 2.990	-8.76933601176'-002 -8.77489676350'-002		1.08865104854'+000 9.11429514177'-001	12.448

γ K	N = 4 RE	$\psi_0 = 1.0$	Type III-general $\pm IM$	$T_m \omega_0$
$B_0 = -K \cdot \pi$		$\delta = 0.05$		
0.894 0.580	-2.15263485672'-001 -2.18366262146'-001		1.25058647461'+000 7.53645254838'-001	4.729
0.700L 0.960	-1.69020572248'-001 -1.20276714322'-001		1.19515210687'+000 8.05926249988'-001	5.981
0.4.8 ! 1.690	-1.21105696839'-001 -1.20677353865'-001		8.61223503227'-001 1.13869936898'+000	8.329
0.300 3.410	-7.27703488774'-002 -7.26325293144'-002		9.16428155608'-001 1.08331524761'+000	13.784

γ	$N = 4$	$\Psi_{\sigma} = 1.0$	Type III-general		
K	RE		$\pm IM$	$T_m \omega_0$	
$B_0 = -K \cdot \pi$			$\delta = 0.07$		
0.894	-1.89401987969'	-001	7.62419978109'	-001	5.043
0.670	-1.87258028484'	-001	1.24088456435'	+000	
0.697	-1.47280243444'	-001	8.13760371078'	-001	6.414
1.090	-1.46407815362'	-001	1.18708515412'	+000	
0.498	-1.05242122227'	-001	8.66572646977'	-001	8.910
1.870	-1.05070168414'	-001	1.13377195294'	+000	
0.299	-6.32145091913'	-002	9.19716110744'	-001	14.766
3.720	-6.31924953523'	-002	1.08032747143'	+000	

γ K	N= 4 $\psi_o=1.0$		Type III-general	
	RE		$\pm IM$	$T_m \omega_b$
	$B_o = - K \cdot \pi$		$\delta = 0.1$	
0.897 0.770	-1.59417592291'-001 -1.60643550429'-001		1.23251137591'+000 7.70542983879'-001	5.388
0.698 1.230	-1.24775199194'-001 -1.24341130544'-001		8.20658911209'-001 1.18039318017'+000	6.876
0.500 2.070	-8.92717320676'-002 -8.91606086516'-002		8.71277019323'-001 1.12898749416'+000	9.555
0.300 4.070	-5.35361234943'-002 -5.35219933834'-002		9.22700311501'-001 1.07733295282'+000	15.875

γ K	N = 6 $\psi_0 = 1.0$		Type III-general	
	RE		$\pm IM$	$T_m \omega_0$
	$B_0 = -K \cdot \pi$		$\delta = 0.01$	
0.895	-2.19749089541'-001		1.37047933252'+000	6.232
0.610	-2.25474140199'-001		6.35704390003'-001	
	-2.62572971315'-001		1.00552230672'+000	
0.697	-1.74522197976'-001		7.13643085644'-001	7.876
1.100	-1.72539071406'-001		1.28782087669'+000	
	-2.05231032927'-001		1.00161837065'+000	
0.499	-1.24519178069'-001		7.94393343193'-001	10.888
2.030	-1.24023753597'-001		1.20596015087'+000	
	-1.47124837153'-001		1.00040544011'+000	
0.300	-7.44996239337'-002		1.12346976448'+000	18.040
4.280	-7.45615354426'-002		8.76573543756'-001	
	-8.82623760845'-002		1.00005077198'+000	

γ K	N = 6 $\psi_0 = 1.0$		Type III-general	
	RE		$\pm IM$	$T_m \omega_0$
	$B_0 = -K \cdot \pi$		$\delta = 0.03$	
0.898	-1.63153687325'-001		1.33906173171'+000	7.193
0.880	-1.65548120243'-001		6.64292528090'-001	
	-1.88810003230'-001		1.00291111897'+000	
0.698	-1.27988851644'-001		7.38508731545'-001	9.181
1.490	-1.27173358578'-001		1.26290591416'+000	
	-1.46767891376'-001		1.00114147074'+000	
0.499	-1.04973325826'-001		1.00040967076'+000	12.764
2.610	-9.11021127104'-002		1.18781495810'+000	
	-9.13053170807'-002		8.12783530913'-001	
0.300	-5.48189095721'-002		1.11287678138'+000	21.149
5.260	-5.48446508183'-002		8.87381153207'-001	
	-6.31191769753'-002		1.00014313334'+000	

γ K	N = 6 RE	$\psi_0 = 1.0$	Type III-general $\pm IM$	$T_m \omega_0$
$B_0 = -K \cdot \pi$		$\delta = 0.05$		
0.899	-1.37721527161'-001		1.32559965947'+000	7.715
1.030	-1.39449334962'-001		6.75861415832'-001	
	-1.56619956961'-001		1.00159097445'+000	
0.700	-1.07476701309'-001		1.25329623558'+000	9.846
1.690	-1.07993736747'-001		7.47390388104'-001	
	-1.21952592192'-001		1.00065243756'+000	
0.499	-8.70502358812'-002		1.00016143045'+000	13.733
2.910	-7.68096343054'-002		1.18062811176'+000	
	-7.69376838071'-002		8.19539049942'-001	
0.300	-4.61734978992'-002		8.91578249863'-001	22.800
5.780	-5.22821868562'-002		1.00002037396'+000	
	-4.61573312914'-002		1.10844264232'+000	
γ K	N = 6 RE	$\psi_0 = 1.0$	Type III-general $\pm IM$	$T_m \omega_0$
$B_0 = -K \cdot \pi$		$\delta = 0.07$		
0.899	-1.21234222712'-001		1.31760425353'+000	8.090
1.140	-1.22483429415'-001		6.83657311049'-001	
	-1.36133517612'-001		1.00131344481'+000	
0.699	-1.05969321252'-001		1.00035510445'+000	10.341
1.840	-9.50404173927'-002		7.53419379849'-001	
	-9.45321324216'-002		1.24681333883'+000	
0.500	-6.76597687962'-002		1.17633375089'+000	14.409
3.120	-6.78517155466'-002		8.23565031285'-001	
	-7.57714621686'-002		1.00000842940'+000	
0.300	-4.06002168993'-002		8.94248654447'-001	23.971
6.150	-4.05886795568'-002		1.10576882603'+000	
	-4.54287423927'-002		1.00001649791'+000	

γ K	N=6 RE	$\psi_0=1.0$	Type III-general $\pm IM$	$T_m \omega_0$
	$B_0 = -K \cdot \pi$		$\delta = 0.1$	
0.898 1.270	-1.03859454954'-001 -1.04714852896'-001 -1.14967168076'-001		1.30892855732'+000 6.92113826433'-001 1.00104412664'+000	8.530
0.699 2.010	-8.14299831218'-002 -8.10712923340'-002 -8.96219179849'-002		7.59724318847'-001 1.24049087084'+000 1.00029478542'+000	10.898
0.500 3.370	-5.80984881706'-002 -5.79576135342'-002 -6.40206923708'-002		8.28262099163'-001 1.17167465034'+000 1.00001728896'+000	15.210
0.300 6.570	-3.47873767532'-002 -3.48426002893'-002 -3.84127671025'-002		1.10297703632'+000 8.96930921819'-001 9.99961412977'-001	25.301

γ K	N=8 RE	$\psi_0=1.0$	Type III-general $\pm IM$	$T_m \omega_0$
	$B_0 = -K \cdot \pi$		$\delta = 0.01$	
0.897 1.030	-1.53981116473'-001 -1.57301955012'-001 -1.90114131372'-001 -1.91927361531'-001		1.39510527888'+000 6.06557328109'-001 1.13677501335'+000 8.67159305315'-001	9.141
0.699 1.790	-1.20473533625'-001 -1.21767568959'-001 -1.48463902639'-001 -1.49399144575'-001		1.30755848408'+000 6.92904374526'-001 1.10555750714'+000 8.99254381012'-001	11.625
0.500 3.200	-8.64078769737'-002 -1.06848926524'-001 -1.06362269959'-001 -8.68697015212'-002		1.21999027545'+000 9.24600404978'-001 1.07516213087'+000 7.80060107500'-001	16.142

γ K	N=8 RE	$\psi_0=1.0$	Type III-general $\pm IM$	$T_m \omega_0$
		$B_0 = -K \cdot \pi$	$\delta = 0.03$	
0.900 1.400	-1.15326839841'-001 -1.16579932816'-001 -1.37133644548'-001 -1.37605997388'-001		1.36597115954'+000 6.35711240451'-001 1.12687096605'+000 8.76538569100'-001	10.406
0.698 2.250	-1.08291264234'-001 -9.10251013120'-002 -1.08116851479'-001 -9.14728600692'-002		9.02108140966'-001 1.28796169238'+000 1.09932391532'+000 7.12838388805'-001	13.142
0.500 3.860	-7.75675063705'-002 -6.53137118243'-002 -6.54255200352'-002 -7.75229561841'-002		9.29458297851'-001 1.20612068643'+000 7.93981489725'-001 1.07080423669'+000	18.266

γ K	N=8 RE	$\psi_0=1.0$	Type III-general $\pm IM$	$T_m \omega_0$
		$B_0 = -K \cdot \pi$	$\delta = 0.05$	
0.894 1.550	-9.91460883888'-002 -9.99939390313'-002 -1.15390857216'-001 -1.15704997210'-001		1.35873756500'+000 6.42615498102'-001 1.12397560896'+000 8.78700381614'-001	10.920
0.696 2.500	-7.73336723469'-002 -7.76229671174'-002 -9.00327347918'-002 -8.99220978026'-002		1.27908617660'+000 7.21516104997'-001 9.05022779130'-001 1.09604065924'+000	13.962
0.499 4.221	-5.54745183997'-002 -5.55469349201'-002 -6.44695562951'-002 -6.44978283963'-002		1.19992136222'+000 8.00351208658'-001 1.06864570128'+000 9.31743903086'-001	19.422

γ	$N=8$	$\psi_0=1.0$	Type III-general	
K	RE		$\pm IM$	$T_m \omega_0$
	$B_0 = -K \cdot \pi$		$\delta = 0.1$	
0.891	-7.57396280387'-002		1.34391699883'+000	11.858
1.830	-7.61616027448'-002		6.56804946691'-001	
	-8.53730445087'-002		8.83479063064'-001	
	-8.52211091260'-002		1.11806603108'+000	
0.694	-6.65126965978'-002		9.08835676411'-001	15.168
2.870	-5.91010843582'-002		1.26788619283'+000	
	-6.64588728541'-002		1.09169750288'+000	
	-5.92464663455'-002		7.32358504207'-001	
0.599	-5.09888815677'-002		1.23098920565'+000	17.567
3.628	-5.10672375967'-002		7.69141927638'-001	
	-5.73249734607'-002		1.07899803016'+000	
	-5.73542788661'-002		9.21289535239'-001	

Lebenslauf

Ich wurde am 24.8.1945 in Nord-Ägypten (El-arich) geboren. Nach Besuch der Volksschule und Vorschule trat ich in die höhere Schule ein. Dort schloß ich im Juli 1962 mit dem Abitur ab.

Im Oktober 1962 begann ich an der Ain-shams Universität in Kairo (Fachrichtung Elektrotechnik) und ich schloß den B. Sc. Grad (Ingenieur Grad) im Juni 1967 ab.

Danach arbeitete ich bei der Ain-shams Universität (Nachrichtentechnische Abteilung) als Assistent und begann gleichzeitig ein Forschungsprogramm für den akademischen Grad M. Sc. (Fachrichtung: Theoretische Nachrichtentechnik).

Im Oktober 1970 erhielt ich diesen Grad. In diesem Jahr danach habe ich ein Stipendium des Deutschen Akademischen Austauschdienstes (DAAD) bekommen.

Seit Oktober 1972 bin ich am Lehrstuhl für Nachrichtentechnik der Ruhr-Universität - Bochum tätig.

